# The Discrete Voronoi Game in a Simple Polygon

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**Abstract.** Let $P$ be a simple polygon with $m$ vertices and let $\mathcal{U}$ be a set of $n$ points in $P$. We consider the points of $\mathcal{U}$ to be "users". We consider a game with two players $\mathcal{P}_1$ and $\mathcal{P}_2$. In this game, $\mathcal{P}_1$ places a point facility inside $P$, after which $\mathcal{P}_2$ places another point facility inside $P$. We say that a user $u \in \mathcal{U}$ is served by its nearest facility, where distances are measured by the geodesic distance in $P$. The objective of each player is to maximize the number of users they serve. We show that for any given placement of a facility by $\mathcal{P}_1$, an optimal placement for $\mathcal{P}_2$ can be computed in $O(m + n(\log n + \log m))$ time. We also provide a polynomial-time algorithm for computing an optimal placement for $\mathcal{P}_1$.

## 1 Introduction

In a facility location problem, we are interested in finding a placement of a set of facilities so that, for a given set of users, certain optimality criteria are met. In a typical geometric facility location problem, the facilities and users are modeled as points. Each user is served by its *nearest* facility, with respect to an appropriate distance measure (e.g., Euclidean distance). Consequently, each facility has its *service zone*, consisting of the set of users that are served by it. The aim is to place the facilities so that certain optimality criteria are satisfied.

The *Voronoi game* is a competitive facility location problem introduced by Ahn et al.[1]. Given a user space, two players, $\mathcal{P}_1$ and $\mathcal{P}_2$, sequentially place a set of point facilities. These facilities partition the user space into a set of regions, such that all users within a region are served by a particular facility. The objective of each player is to maximize the total service zone of all its facilities. This problem is generally intractable. Teramoto, Demaine, and Uehara [8] have shown that even if the underling user space is a graph, finding a winning strategy of $\mathcal{P}_2$ (even for a very restricted case) is NP hard. Similar results can also be found in a seminal paper of Hakimi [7].

The discrete version of the Voronoi game is studied by Banik, Battacharya and Das [3]. Their user space is a line containing a set $\mathcal{U}$ of $n$ point users. Each of the players $\mathcal{P}_1$ and $\mathcal{P}_2$ can place $k = O(1)$ point facilities. First, $\mathcal{P}_1$ chooses a set $F_1$ of $k$ facilities, after which $\mathcal{P}_2$ chooses a set $F_2$ of $k$ facilities, disjoint from $F_1$. The *payoff* of $\mathcal{P}_2$ is defined to be the cardinality of the set of points in $\mathcal{U}$ which are closer to some facility owned by $\mathcal{P}_2$ than to every facility owned by $\mathcal{P}_1$. The payoff of $\mathcal{P}_1$ is the number of users in $\mathcal{U}$ minus the payoff of $\mathcal{P}_2$. The objective of both players is to maximize their respective payoffs. Banik et al. show that, if the sorted order of points in $\mathcal{U}$ along the line is given, an optimal strategy of $\mathcal{P}_2$ for any given placement of facilities of $\mathcal{P}_1$ can be computed in linear time. They also provide results for determining an optimal strategy for $\mathcal{P}_1$.

Given a set of existing facilities, the problem of placing a set of new facilities, to maximize the number of users served by the new ones, has been actively researched. Cabello et al. [5] study the case when only one new facility by $\mathcal{P}_2$ is introduced. This problem is referred to as the MaxCov problem. They have shown that the optimal placement for the new facility can be found in $O(n^2)$ time. The 2-MaxCov problem, which considers the problem of placing two new facilities, has been studied by Bhattacharya and Nandy [4]. Recently Bandyapadhyay, Banik, Das and Sarkar [2] studied the one round discrete Voronoi game for graphs.

In this paper, we consider the Voronoi game, where the underlying user space is a simple polygon $P$ with distance measure defined to be the geodesic (i.e., shortest-path) distance in $P$. The game consists of a set $\mathcal{U}$ of $n$ point-users inside $P$, and two players $\mathcal{P}_1$ and $\mathcal{P}_2$. Initially, $\mathcal{P}_1$ places a set $F_1$ of $k$ point-facilities, after which $\mathcal{P}_2$ places a set $F_2$ of $k$ point-facilities, where $F_1 \cap F_2 = \emptyset$. Each user $u \in \mathcal{U}$ is served by the nearest facility according to the nearest neighbor rule (i.e., by the facility which is at the least geodesic distance from $u$).

**Definition 1.** *(Service zone) For each facility $f \in F_1 \cup F_2$, we define its* service zone $\mathcal{U}_{F_1 \cup F_2}(\{f\})$ *to be the set of users in $\mathcal{U}$ that are closer to $f$ than to any other facility of $F_1 \cup F_2$.*

*Given a set $S \subseteq F_1 \cup F_2$, we define the service zone of $S$ to be the set of users which are assigned to one the facilities in $S$, i.e., $\mathcal{U}_{F_1 \cup F_2}(S) = \cup_{f \in S} \mathcal{U}_{F_1 \cup F_2}(\{f\})$.*

With this definition, the problem considered in this paper can be formally described as follows.

**Definition 2.** Discrete Voronoi Game for a Simple Polygon $P$: *Given a set $\mathcal{U}$ of $n$ point-users and two players $\mathcal{P}_1$ and $\mathcal{P}_2$, having $k$ facilities each, $\mathcal{P}_1$ chooses a set $F_1$ of $k$ point-facilities in $P$, after which $\mathcal{P}_2$ chooses a set $F_2$ of $k$ point-facilities in $P$, where $F_1 \cap F_2 = \emptyset$.*

*(a) Given any choice of $F_1$ by $\mathcal{P}_1$, the objective of $\mathcal{P}_2$ is to choose a set $S = F_2$ that maximizes $\mathcal{U}_{F_1 \cup S}(S)|$ over all sets $S$, with $|S| = k$ and $F_1 \cap S = \emptyset$.*
*(b) The objective of $\mathcal{P}_1$ is to place a set $F_1$ of $k$ facilities such that the maximum possible payoff of $\mathcal{P}_2$ is minimized. In other words, the objective of $\mathcal{P}_1$ is to choose a set $F = F_1$ of size $k$ that minimizes $\max_S |\mathcal{U}_{F \cup S}(S)|$, where the maximum is taken over all sets $S$, with $|S| = k$ and $F \cap S = \emptyset$.*

In this paper, we consider the case when $k = 1$. Thus, $\mathcal{P}_1$ will place a single facility inside $P$, after which $\mathcal{P}_2$ places another facility inside $P$. In the next section, we characterize an optimal placement for $\mathcal{P}_2$ and show that, given any placement of a facility by $\mathcal{P}_1$, an optimal strategy for $\mathcal{P}_2$ can be computed in $O(m+n(\log n+\log m))$ time, where $m$ is the number of vertices of $P$. In Section 3, we will provide an algorithm that computes an optimal strategy for $\mathcal{P}_1$.

## 2 Computing an Optimal Placement for $\mathcal{P}_2$

Let $f$ (= first) and $s$ (= second) be the facilities in $P$ that are placed by players $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively.

Given any placement of the facility $f$ by $\mathcal{P}_1$, we will provide an algorithm that computes a point $s$ that maximizes $|\mathcal{U}_{\{f,s\}}(\{s\})|$ over all points $s \in P$, where $s \neq f$.

Consider the set $\mathcal{U} = \{u_1, u_2, \ldots, u_n\}$ of users. For each user $u_i$, let $d_i$ denote the geodesic distance between $u_i$ and $f$, and let $\Gamma_i$ denote the set of points in the polygon $P$ whose geodesic distance to $u_i$ is at most $d_i$ (see Figure 1).

**Observation 1** *A user $u_i \in \mathcal{U}$ is served by a facility $s$ placed by $\mathcal{P}_2$ if and only if $s$ belongs to $\Gamma_i$.*

Hence, if we consider the arrangement inside $P$ defined by the set of regions $\Gamma = \{\Gamma_1, \Gamma_2, \ldots \Gamma_n\}$, then an optimal placement for $\mathcal{P}_2$ belongs to a cell in this arrangement having maximum depth.

For any two points $p_1$ and $p_2$ in $P$, denote the geodesic path between $p_1$ and $p_2$ by $\lambda(p_1, p_2)$. The length of a geodesic path $\lambda$ is denoted by $|\lambda|$. The *anchor* of $f$ with respect to $u_i$ is defined to be the last vertex on the path $\lambda(u_i, f)$ from $u_i$ to $f$; we denote this anchor by $a_i(f)$. If $f$ is visible from $u_i$ then we define $a_i(f) = u_i$. Let $C_i$ denote the circle centered at $a_i(f)$ and passing through $f$. For any two points $p_1, p_2 \in P$, denote the line segment joining them by $[p_1, p_2]$.

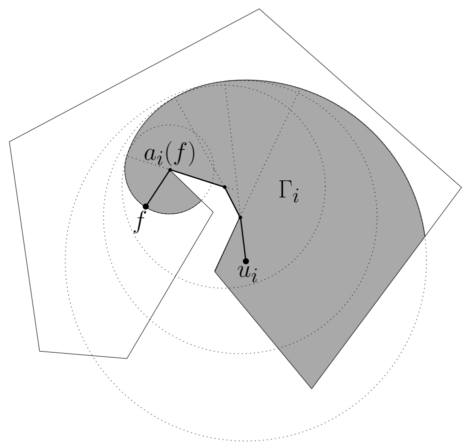


**Fig. 1.** Span of the user $u_i$

For any anchor vertex $a_i(f)$, let $l_i$ denote the line tangent to the circle $C_i$ and passing through the point $f$ (see Figure 2(a)). Consider the line segment $[c, d] \subset l_i$of maximum length that is completely contained in $P$ and that contains $f$. Observe that $[c, d]$ divides the polygon into two parts. Denote the part which contains $u_i$ by $P_i$.

**Lemma 1.** *For any user $u_i$, $\Gamma_i \subseteq P_i$.*

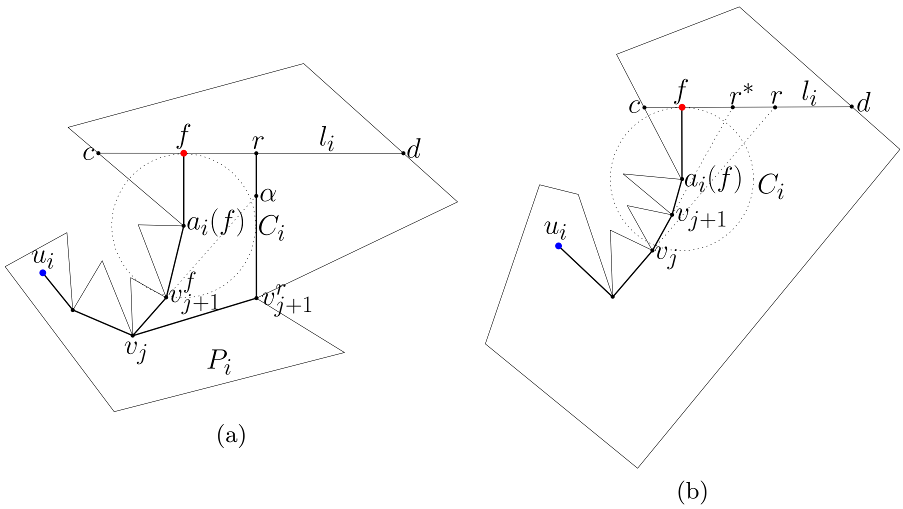


**Fig. 2.** Illustration of the proof of Lemma 1

*Proof.* If $a_i(f) = u_i$, then there is nothing to prove. In the rest of the proof, we assume that $a_i(f) \neq u_i$. It is sufficient to prove that for any point $q \in [c, d]$, $|\lambda(u_i, q)| > |\lambda(u_i, f)|$. Observe that $[a_i(f), f]$ divides $P_i$ in two parts. Without loss of generality, assume that $d$ and $u_i$ belong to the same sub-polygon and that $c$ belongs to the other sub-polygon. For all points $q \in [c, f]$, the shortest path from $u_i$ to $q$ is through $a_i(f)$. Hence, for all points $q \in [c, f]$, $|\lambda(u_i, q)| > |\lambda(u_i, f)|$. Assume there exists a point $r$ in $[f, d]$, such that $|\lambda(u_i, r)| < |\lambda(u_i, f)|$. Let $r \in [f, d]$ be such a point that is closest to $f$.

**Claim:** The set of vertices in $\lambda(u_i, r)$ is a subset of the set of vertices in $\lambda(u_i, f)$.
*Proof of Claim.* Let the vertices on the path $\lambda(u_i, f)$ be

$$\lambda(u_i, f) = (v_1, v_2, \ldots, v_j, v^f_{j+1}, \ldots, v^f_\tau, f)$$

and let the vertices on the path $\lambda(u_i, r)$ be

$$\lambda(u_i, r) = (v_1, v_2, \ldots, v_j, v^r_{j+1}, \ldots, v^r_\omega, r),$$

see Figure 2(a). Observe that the line joining $v_j$ and $v_{j+1}^f$ either intersects the line segment $[f,r]$ or intersects some edge in $\lambda(v_j,r)$. If this line intersects $[f,r]$ then that contradicts the fact that $r$ is the closest point from $f$ in $[f,d]$ for which $|\lambda(u_i,r)| < |\lambda(u_i,f)|$. Hence, the line joining $v_j$ and $v_{j+1}^f$ intersects an edge of $\lambda(v_j,r)$ (see Figure 2(a)) at a point $\alpha$. From the convexity properties of geodesic paths and the triangle inequality, $|\lambda(v_j,v_{j+1}^f,\alpha)| < |\lambda(v_j,v_{j+1}^r,\alpha)|$. This contradicts the fact that the shortest path between $v_j$ and $r$ is via $v_{j+1}^r$. Hence the claim holds.

We continue with the proof of Lemma 1. Let $j$ be the index such that $r$ is the intersection of $l_i$ and the line joining the two consecutive vertices $v_j$ and $v_{j+1}$ of $\lambda(u_i,f)$ (see Figure 2(b)). Denote the intersection between $l_i$ and the line joining $v_{j+1}$ and $v_{j+2}$ by $r^*$. Observe that $|\lambda(u_i,r^*)| < |\lambda(u_i,r)|$. This contradicts the fact that $r$ is the closest point from $f$ in $[f,d]$ for which $|\lambda(u_i,r)| < |\lambda(u_i,f)|$. Hence $r$ must be the intersection of $l_i$ and the line joining $a_i(f)$ and $b_i$, where $b_i$ is the vertex previous to $a_i(f)$ on the path $\lambda(u_i,f)$. But $|\lambda(a_i(f),f)| < |\lambda(a_i(f),r)|$. Therefore, $|\lambda(u_i,f)| < |\lambda(u_i,r)|$. Hence, we arrive at a contradiction. □

For any anchor vertex $a_i(f)$, let $Z_i$ denote the set of points in $P$ which are at distance at most $|a_i(f)f|$ from $a_i(f)$.

**Observation 2** *For any two users $u_i$ and $u_j$, we have $(\mathit{\Gamma}_i \cap \mathit{\Gamma}_j) \setminus \{f\} = \emptyset$ if and only if $(Z_i \cap Z_j) \setminus \{f\} = \emptyset$.*

*Proof.* Observe that $Z_i \subseteq \mathit{\Gamma}_i$ and $Z_j \subseteq \mathit{\Gamma}_j$. Therefore, if $(Z_i \cap Z_j) \setminus \{f\} \neq \emptyset$, then $(\mathit{\Gamma}_i \cap \mathit{\Gamma}_j) \setminus \{f\} \neq \emptyset$.

Assume that $(Z_i \cap Z_j) \setminus \{f\} = \emptyset$. Observe that both $Z_i$ and $Z_j$ contain the point $f$. Hence, both the circles $C_i$ and $C_j$ share the same tangent $l_{ij}$ passing through $f$. Now $l_{ij}$ divides $P$ into two disjoint subpolygons $P_i$ and $P_j$. Observe that $P_i \subset \mathit{\Gamma}_i$ and $P_j \subset \mathit{\Gamma}_j$. Therefore, $(\mathit{\Gamma}_i \cap \mathit{\Gamma}_j) \setminus \{f\} = \emptyset$. □

**Observation 3** *For any placement of a facility $f$ by $\mathcal{P}_1$, an optimal placement of a facility by $\mathcal{P}_2$ is the point $\mathring{s} \neq f$ in $P$ that pierces the maximum number of regions among $\{Z_1, Z_2, \ldots Z_n\}$.*

The arrangement of the regions $Z_1, Z_2, \ldots Z_n$ divides $P$ into cells. All points within the same cell pierce the same set of regions. Define the *depth* of a cell to be the number of regions pierced by any point in that cell. The cell with maximum depth contains $f$, because all regions $Z_1, Z_2, \ldots Z_n$ contain $f$.

Consider a circle $C_\epsilon$ with radius $\epsilon > 0$ that is centered at $f$; this circle pierces all cells containing $f$ (see Figure 3(a)). Observe that $C_\epsilon \cap P$ can be a set of disjoint subsets of $C_\epsilon$. If $f$ is in the interior of $P$, then we can choose $\epsilon$ such that $C_\epsilon$ is completely contained in the interior of $P$. If $f$ belongs to the boundary of $P$, then we can choose $\epsilon$ such that $C_\epsilon \cap P$ is a single connected subset of $C_\epsilon$ (see Figure 3(a) where $C_\epsilon \cap P$ consists of three disjoint sets $\mu_1, \mu_2$, and $\mu_3$).

Consider any optimal placement $s$ for $\mathcal{P}_2$. Let $\gamma$ be the cell that contains $s$. From the previous discussion, any point in $\gamma$ acts as an optimal placement for

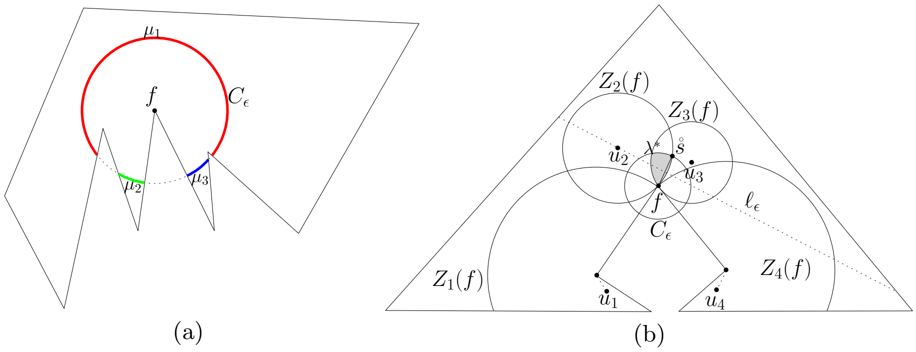


**Fig. 3.** Arrangement of the regions $\{Z_1, Z_2, \ldots Z_n\}$

$\mathcal{P}_2$. Hence, the intersection point between the boundary of $\gamma$ and $C_\epsilon$ is also an optimal placement. Thus, one of the optimal placements for $\mathcal{P}_2$ belongs to the set $\alpha_\epsilon = \{C_i \cap C_\epsilon : 1 \le i \le n\}$.

Consider any optimal placement of facility $\mathring{s} \in \alpha_\epsilon$ for $\mathcal{P}_2$. Let $\mathring{s}$ be the intersection point between $C_i$ and $C_\epsilon$. Consider the perpendicular bisector $\ell$ of $f$ and $\mathring{s}$. Let $\ell_\epsilon \subseteq \ell$ be the maximal line segment in $P$ that contains the midpoint of the line segment joining $f$ and $\mathring{s}$ (see Figure 3(b)).

**Observation 4** *The line segment $\ell_\epsilon$ passes through $a_i(f)$.*

*Proof.* Observe that $[\mathring{s}, f]$ is a chord of the circle $C_i$. The perpendicular bisector of any chord always passes through the center of the circle. Hence, the result holds. □

Note that $\ell_\epsilon$ divides $P$ into two sub-polygons, one containing $f$ and the other containing $\mathring{s}$. If $\mathcal{P}_2$ places its facility at $\mathring{s}$, $\mathcal{P}_2$ will serve the set of users $u_i$ such that $a_i(f)$ belongs to the sub-polygon containing $\mathring{s}$. As $\epsilon$ tends to 0, $\ell_\epsilon$ tends to the line joining $f$ and $a_i(f)$. Hence, for all $a_i(f)$, if we consider the chord passing through $f$ and $a_i(f)$, then we can find an optimal placement for $\mathcal{P}_2$. Note that all anchor vertices are visible from $f$. Thus, using an angular sorting, we can find an optimal placement for $\mathcal{P}_2$. We obtain the following result.

**Theorem 1.** *Let $P$ be a polygon with $m$ vertices and let $\mathcal{U}$ be a set of $n$ point-users in $P$. Given the placement of a point $f \in P$ by $\mathcal{P}_1$, a point $s \in P$ maximizing $\mathcal{P}_2$'s payoff can be computed in $O(m + n(\log m + \log n))$ time.*

*Proof.* Let $f$ be any placement of a facility by $\mathcal{P}_1$. Consider the visibility region $V_f$ of $f$ in $P$, i.e., the set of points which are visible from $f$. Observe that $P \setminus V_f$ consists of a set of possibly disjoint sub-polygons of $P$. For each such sub-polygon $P_i$, for all points $q \in P_i$, the anchor vertex on the path $\lambda(q, f)$ will be the same. Given $f$, we can construct a data structure in $O(m)$ time that can report the anchor vertex on the path $\lambda(q, f)$, for any query point $q \in P$, in $O(\log m)$ time [6]. Using this data structure, in $O(n \log m)$ time, we can find the set of all anchor vertices on the paths from users in $\mathcal{U}$ to $f$. Once we have the list of anchor vertices, using angular sorting, we can compute the half plane passing through $f$ which contains the maximum number of anchor vertices. Hence the result follows. □

# 3 Computing an Optimal Placement for $\mathcal{P}_1$

As before, let $P$ be a simple polygon with $m$ vertices and let $\mathcal{U} = \{u_1, u_2 \dots u_n\}$ be a set of $n$ point-users in $P$. We will present an algorithm that computes an optimal placement of a facility for $\mathcal{P}_1$.

For any placement of $f$ by $\mathcal{P}_1$, let $\nu(f) = \max_s |\mathcal{U}_{f \cup s}(\{s\})|$, where the maximum is taken over all points $s$ in $P$ with $s \neq f$. Our objective is to find a point $\mathring{f} \in P$ which minimizes $\nu$; we call such a point an *optimal placement* for $\mathcal{P}_1$. Observe that there are two cases:

*Case 1:* $\mathring{f}$ belongs to the boundary of $P$.
*Case 2:* $\mathring{f}$ is in the interior of $P$.

In Section 3.1, we will give an algorithm that computes an optimal placement on the boundary of $P$ for $\mathcal{P}_1$. Formally, we will show how to compute a point $f_b$ on the boundary of $P$ such that $\nu(f_b) = \min_f \nu(f)$, where the minimum is taken over all points $f$ on the boundary of $P$. In Section 3.2, we will give an algorithm that computes an optimal placement in the interior of $P$ for $\mathcal{P}_1$.

## 3.1 The boundary case

Let us begin our discussion with the following two simple observations (see Figure 4).

**Observation 5** *For any placement $f$ by $\mathcal{P}_1$, where $f$ is on any convex vertex of $P$, there exists a placement $s$ for $\mathcal{P}_2$ such that $\nu(f) = n$.*

**Observation 6** *Let $(v_i, v_{i+1})$ be an edge of $P$ such that at least one of $v_i$ and $v_{i+1}$ is a convex vertex. For any placement $f$ by $\mathcal{P}_1$ on the edge $(v_i, v_{i+1})$, there exists a placement $s$ for $\mathcal{P}_2$ such that $\nu(f) = n$.*

Hence, an optimal placement for $\mathcal{P}_1$ must be either at a reflex vertex or on an edge $(v_i, v_{i+1})$ for which both $v_i$ and $v_{i+1}$ are reflex vertices.

**Observation 7** *Let $(v_i, v_{i+1})$ be an edge of $P$ such that both $v_i$ and $v_{i+1}$ are reflex vertices. For any placement $f$ by $\mathcal{P}_1$ on the edge $(v_i, v_{i+1})$, $\nu(f) \geq \nu(v_i)$ and $\nu(f) \geq \nu(v_{i+1})$.*

*Proof.* Let $p$ and $q$ be arbitrary placements by $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. The perpendicular bisector of $p$ and $q$ divides $P$ into two sub-polygons. Denote the sub-polygon that contains $p$ by $P^+(p,q)$, and the other sub-polygon by $P^-(p,q)$.

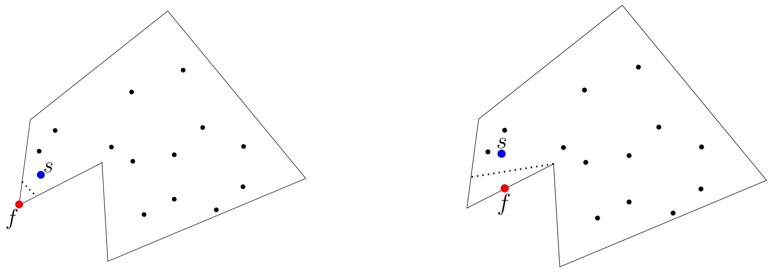


**Fig. 4.** (a) Illustration of Observation 5 (b) Illustration of Observation 6

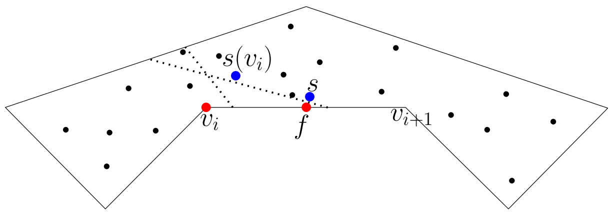


**Fig. 5.** Illustration of the proof of Observation 7

Let $f$ be any placement by $\mathcal{P}_1$ on the edge $(v_i, v_{i+1})$ (see Figure 5). Let $s(v_i)$ be an optimal placement for $\mathcal{P}_2$, when $\mathcal{P}_1$ places its facility at $v_i$. Hence, $\nu(v_i)$ is the number of users in $P^-(v_i, s(v_i))$. When $\mathcal{P}_1$ places its facility at $f$, there always exists a placement $s$ by $\mathcal{P}_2$, such that $s$ serves the set of users in $P^-(v_i, s(v_i))$ (see Figure 5). Therefore, $\nu(f) \geq \nu(v_i)$ and the claim holds. $\square$

Thus, there is optimal placement, on the boundary of $P$, for $\mathcal{P}_1$ that is at a reflex vertex of $P$. By checking all reflex vertices, we can compute an optimal placement for $\mathcal{P}_1$ on the boundary of $P$ in $O(m^2 \log m)$ time.

### 3.2 The interior case

In this section, we present an algorithm that computes an optimal placement for $\mathcal{P}_1$ in the interior of the polygon $P$. Let $R$ denote the set of reflex vertices of $P$. Consider the set $L$ of all maximal line segments which are fully contained in $P$ and contain at least two points from $R \cup \mathcal{U}$ (see Figure 6(a)). The set $L$ tessellates $P$ into a collection of cells. Denote the tessellation by $\Pi(P)$.

Recall the notion of an anchor vertex defined in Section 2.

**Lemma 2.** *For any cell $C$ in $\Pi(P)$, for any two points $f_1$ and $f_2$ in $C$, and for any user $u_i$, we have $a_i(f_1) = a_i(f_2)$.*

*Proof.* Assume there exists a user $u_i$ whose anchor vertex $a_i(f_1)$ on the geodesic path from $u_i$ to $f_1$ is different from the anchor vertex $a_i(f_2)$ on the geodesic path from $u_i$ to $f_2$.

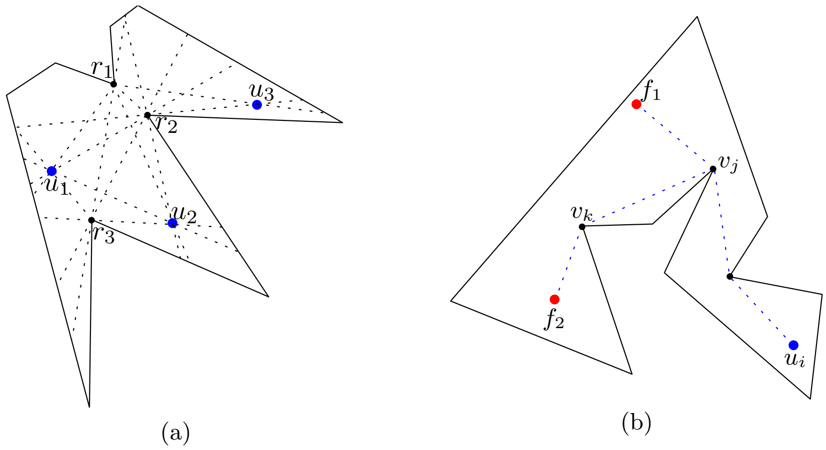


**Fig. 6.** (a) Tessellation of $P$ (b) Illustration of the proof of Lemma 2

Let $v_j$ be the last vertex that is common to the paths $\lambda(u_i, f_1)$ and $\lambda(u_i, f_2)$. Observe that one of $a_i(f_1)$ and $a_i(f_2)$ is not equal to $v_j$, because otherwise, we would have $a_i(f_1) = a_i(f_2) = v_j$. Assume, without loss of generality, that $a_i(f_2) \neq v_j$ (see Figure 6(b)).

Let $v_k$ be the vertex next to $v_j$ on the shortest path from $v_j$ to $f_2$. Observe that $f_1$ and $f_2$ are on different sides of the line joining $v_j$ and $v_k$. Hence, $f_1$ and $f_2$ belong to two different cells of the tessellation $\Pi(P)$. □

Let $C$ be any cell in $\Pi(P)$. For any user $u_i$, all points $f$ in $C$ have the same anchor vertex; denote this anchor vertex by $a_i^C$. Assign a weight $w_i^C$ which is the number of shortest paths from any user $u_j$ to any point $f \in C$, in which $a_i^C$ is the anchor vertex.

Recall that a chord of $P$ is a closed line segment whose interior is contained in the interior of $P$ and whose endpoints are on the boundary of $P$. Let $f$ be any placement by $\mathcal{P}_1$. Any chord passing through $f$ divides $P$ into two sub-polygons, which we call half polygons with respect to $f$. From Section 2, we know that for any placement $f$ by $\mathcal{P}_1$, the maximum number of users that $\mathcal{P}_2$ can serve, by placing one facility, is equal to $\max_{P_f} \sum_{a_i^C \in P_f} |w_i^C|$, where $P_f$ is any half polygon with respect to $f$, i.e., the maximum number of anchor vertices in any half polygon with respect to $f$.

For any point $f$ in any cell $C$, we define the *weighted half-space depth* of $f$ to be $\max_{P_f} \sum w_j^C$ such that $a_j \in P_f$. Observe that an optimal placement for $\mathcal{P}_1$ in the cell $C$ corresponds to a point with minimum weighted half-space depth.

**Lemma 3.** *One of the optimal placements for $\mathcal{P}_1$ belongs to the set of vertices of the tessellation $\Pi(P)$.*

*Proof.* Assume that none of the optimal placements for $\mathcal{P}_1$ belongs to the set of vertices of $\Pi(P)$. Let $f$ be any optimal placement for $\mathcal{P}_1$. Suppose $f$ belongs to

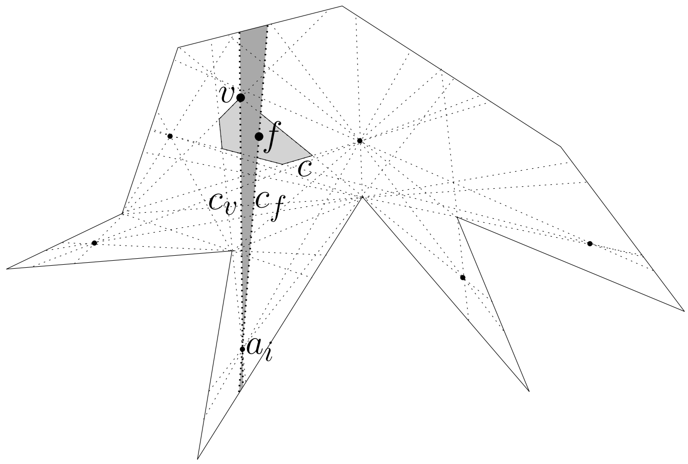


**Fig. 7.** Illustration of the proof of Lemma 3

the cell $C \in \Pi(P)$. Let $v$ any vertex of this cell. Let $\delta$ be the payoff of $\mathcal{P}_1$, when $\mathcal{P}_1$ places a facility at $v$, and assume that $\delta$ is less then the optimal payoff of $\mathcal{P}_1$. Then there exists a half-polygon $P_v$ bounded by a chord $c_v$, which contains $n-\delta$ users. With out loss of generality, we may assume that $c_v$ is passing through some anchor vertex $a_j$. Since $a_j$ is visible from $v$, $a_j$ is also visible from $f$. Consider the chord $c_f$ passing through $f$ and $a_j$ (see Figure 7). Consider the half polygon $P_f$ bounded by $c_f$. Observe that $P_v \setminus P_f = \emptyset$, because otherwise, $v$ and $f$ belong to different cells of $\Pi(P)$. It follows that the claim holds. □

Since the cardinality of $R \cup \mathcal{U}$ is at most $n+m$, the number of cells and the number of vertices in the tessellation $\Pi(P)$ is $O((n+m)^4)$. For each vertex, we can check the optimal payoff of $\mathcal{P}_1$ in $O(m + n(\log n + \log m))$ time. Hence, we have proved the following result.

**Theorem 2.** *Let $P$ be a polygon with $m$ vertices and let $\mathcal{U}$ be a set of $n$ point-users in $P$. An optimal placement of a facility for $\mathcal{P}_1$ can be computed in polynomial time.*

## 4 Conclusion

We have considered the Discrete Voronoi Game for a Simple Polygon $P$. The game consists of two players $\mathcal{P}_1$ and $\mathcal{P}_2$, and a finite set of users in a simple polygon $P$. Initially, $\mathcal{P}_1$ places one facility in $P$, after which $\mathcal{P}_2$ places another facility in $P$. Each user is then assigned to one of the facilities according to the nearest neighbor rule, where distances are measured using the geodesic distance in $P$. We have shown that an optimal strategy for $\mathcal{P}_2$, given any placement of $\mathcal{P}_1$, can be found in $O(m + n(\log m + \log n))$ time, and an optimal strategy for $\mathcal{P}_1$ can be found in polynomial time.

There are many open problems in this area. Obtaining an algorithm to find an optimal placement for $\mathcal{P}_1$ and $\mathcal{P}_2$, where each of them places $k > 1$ facilities is a problem that remains to be solved. Another variant of the game where the two players place $k > 1$ facilities alternately is also an interesting problem to study.

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