Fixed-Orientation Equilateral Triangle Matching of Point Sets

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Abstract

Given a point set P and a class $\mathscr{C}$ of geometric objects, $G_{\mathscr{C}}(P)$ is a geometric graph with vertex set P such that any two vertices p and q are adjacent if and only if there is some $C \in \mathscr{C}$ containing both p and q but no other points from P. We study $G_{\bigtriangledown}(P)$ graphs where $\bigtriangledown$ is the class of downward equilateral triangles (ie. equilateral triangles with one of their sides parallel to the x-axis and the corner opposite to this side below that side). For point sets in general position, these graphs have been shown to be equivalent to half-Θ_6 graphs and TD-Delaunay graphs.

The main result in our paper is that for point sets P in general position, $G_{\bigtriangledown}(P)$ always contains a matching of size at least $\left\lceil \frac{|P|-1}{3} \right\rceil$ and this bound is tight. We also give some structural properties of $G_{\text{✡}}(P)$ graphs, where ✡ is the class which contains both upward and downward equilateral triangles. We show that for point sets in general position, the block cut point graph of $G_{\text{✡}}(P)$ is simply a path. Through the equivalence of $G_{\text{✡}}(P)$ graphs with Θ_6 graphs, we also derive that any Θ_6 graph can have at most $5n-11$ edges, for point sets in general position.

Keywords: Geometric Graphs, Delaunay Graphs, Half-Θ_6 Graphs, Matching

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1. Introduction

In this work, we study the structural properties of some special geometric graphs defined on a set P of n points on the plane. An equilateral triangle with one side parallel to the x-axis and the corner opposite to this side below (resp. above) that side as in $\bigtriangledown$ (resp. $\bigtriangleup$) will be called a down (resp. up)-triangle. A point set P is said to be in general position, if the line passing through any two points from P does not make angles $0°$, $60°$ or $120°$ with the horizontal [1, 2]. In this paper, we consider only point sets that are in general position and our results assume this pre-condition.

Given a point set P, $G_{\bigtriangledown}(P)$ (resp. $G_{\bigtriangleup}(P)$) is defined as the graph whose vertex set is P and that has an edge between any two vertices p and q if and only if there is a down-(resp. up-)triangle containing both points p and q but no other points from P (See Figure 1). We also define another graph $G_{\text{✡}}(P)$ as the graph whose vertex set is P and that has an edge between any two vertices p and q if and only if there is a down-triangle or an up-triangle containing both points p and q but no other points from P. In Section 3 we will see that, for any point set P in general position, its $G_{\bigtriangledown}(P)$ graph is the same as the well known Triangle Distance Delaunay (TD-Delaunay) graph of P and the half-Θ_6 graph of P on so-called negative cones. Moreover, $G_{\text{✡}}(P)$ is the same as the Θ_6 graph of P [1, 3].

Given a point set P and a class $\mathscr{C}$ of geometric objects, the maximum $\mathscr{C}$-matching problem is to compute a subclass $\mathscr{C}'$ of $\mathscr{C}$ of maximum cardinality such that no point from P belongs to more than one element of $\mathscr{C}'$ and for each $C \in \mathscr{C}'$, there are exactly two points from P which lie inside C. Dillencourt [4] proved that every point set admits a perfect circle-matching. Ábrego et al. [5] studied the isothetic square matching problem. Bereg et al. concentrated on matching points using axis-aligned squares and rectangles [6].

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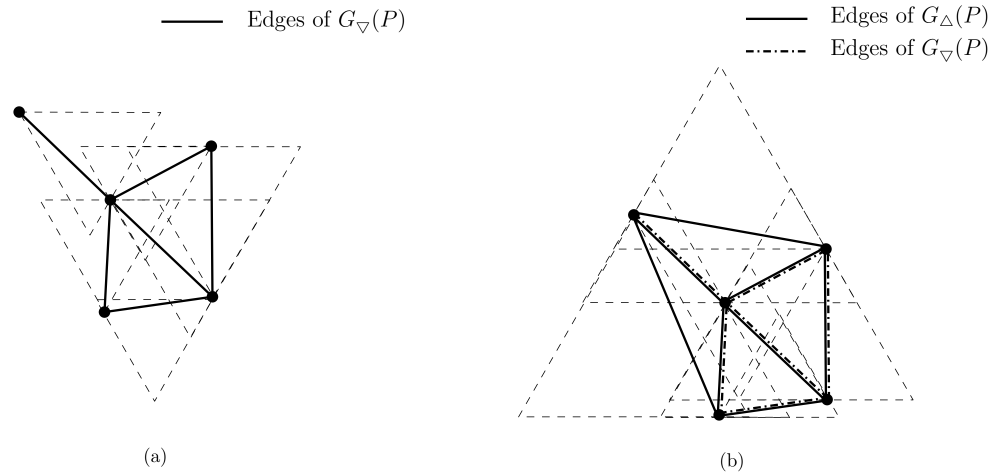


Figure 1: A point set P and its (a) $G_{\bigtriangledown}(P)$ and (b) $G_{✡}(P)$.

A matching in a graph G is a subset M of the edge set of G such that no two edges in M share a common end-point. A matching is called a maximum matching if its cardinality is the maximum among all possible matchings in G. If all vertices of G appear as end-points of some edge in the matching, then it is called a perfect matching. It is not difficult to see that for a class $\mathscr{C}$ of geometric objects, computing the maximum $\mathscr{C}$-matching of a point set P is equivalent to computing the maximum matching in the graph $G_{\mathscr{C}}(P)$.

The maximum $\triangle$-matching problem, which is the same as the maximum matching problem on $G_{\triangle}(P)$, was previously studied by Panahi et al. [2]. It was claimed that, for any point set P of n points in general position, any maximum matching of $G_{\triangle}(P)$ (and $G_{\bigtriangledown}(P)$) will match at least $\left\lfloor \frac{2n}{3} \right\rfloor$ vertices. But we found that their proof of Lemma 7, which is very crucial for their result, has gaps. By a completely different approach, we show that for any point set P in general position, $G_{\bigtriangledown}(P)$ (and by symmetric arguments, $G_{\triangle}(P)$) will have a maximum matching of size at least $\left\lceil \frac{n-1}{3} \right\rceil$; i.e, at least $2\left(\left\lceil \frac{n-1}{3} \right\rceil\right)$ vertices are matched. We also give examples of point sets, where our bound is tight.

We also prove some structural and geometric properties of the graphs $G_{\bigtriangledown}(P)$ (and by symmetric arguments, $G_{\triangle}(P)$) and $G_{✡}(P)$. It will follow that for point sets in general position, Θ_6 graphs can have at most $5n-11$ edges and their block cut point graph is a simple path.

2. Notations

Our notations are similar to those used in [1], with some minor modifications adopted for convenience. A *cone* is the region in the plane between two rays that emanate from the same point, its apex. Consider the rays obtained by a counter-clockwise rotation of the positive x-axis by angles of $\frac{i\pi}{3}$ with $i = 1, \ldots, 6$ around a point p. (See Figure 2). Each pair of successive rays, $\frac{(i-1)\pi}{3}$ and $\frac{i\pi}{3}$, defines a cone, denoted by $A_i(p)$, whose apex is p. For $i \in \{1, \ldots, 6\}$, when i is odd, we denote $A_i(p)$ using $C_{\frac{i+1}{2}}(p)$ and the cone opposite to $C_i(p)$ using $\overline{C}_i(p)$. We call $C_i(p)$ a positive cone around p and $\overline{C_i}(p)$ a negative cone around p. For each cone $\overline{C_i}(p)$ (resp. $C_i(p)$), let $\ell_{\overline{C_i}(p)}$ (resp. $\ell_{C_i(p)}$) be its bisector. If $p' \in \overline{C_i}(p)$, then let $\overline{c_i}(p, p')$ denote the distance between p and the orthogonal projection of p' onto $\ell_{\overline{C_i}(p)}$. Similarly, if $p' \in C_i(p)$, then let $c_i(p, p')$ denote the distance between p and the orthogonal projection of p' onto $\ell_{C_i(p)}$. For $1 \leq i \leq 3$, let $V_i(p) = \{p' \in P \mid p' \in C_i(p), p' \neq p\}$ and $\overline{V_i}(p) = \{p' \in P \mid p' \in \overline{C_i}(p), p' \neq p\}$. For any two points p and q, the smallest down-triangle containing p and q is denoted by $\bigtriangledown pq$ and the smallest up-triangle containing p and q is denoted by $\triangle pq$. If G_1 and G_2 are graphs on the same vertex set, $G_1 \cap G_2$ (resp. $G_1 \cup G_2$) denotes the graph on the same vertex set whose edge set is the intersection (resp. union) of the edge sets of G_1 and G_2.

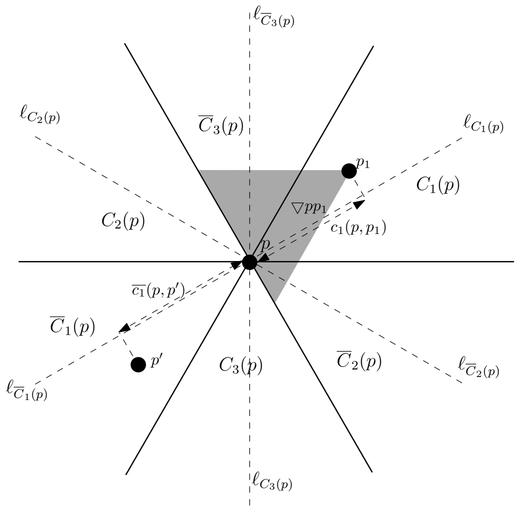


Figure 2: Six angles around a point p.

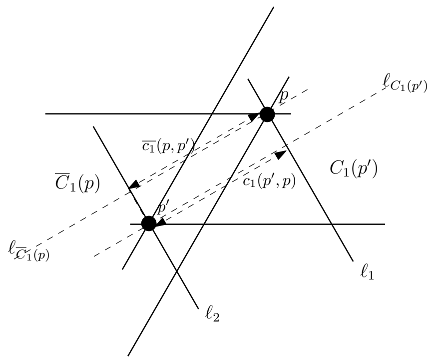


Figure 3: Proof of Property 1.

3. Preliminaries

In this section, we describe some basic properties of the geometric graphs described earlier and their equivalence with other geometric graphs which are well known in the literature.

The class of down-triangles (and up-triangles) admits a shrinkability property [5]: each triangle object in this class that contains two points p and q, can be shrunk such that p and q lie on its boundary. It is also clear that we can continue the shrinking process—from the edge that does not contain neither p or q—until at least one of the points, p or q, becomes a triangle vertex and the other point lies on the edge opposite to this vertex. After this, if we shrink the triangle further, it cannot contain p and q together. Therefore, for any pair of points p and q, $\bigtriangledown pq$ ($\bigtriangleup pq$) has one of the points p or q at a vertex of $\bigtriangledown pq$ ($\bigtriangleup pq$) and the other point lies on the edge opposite to this vertex. In Figure 1, triangles are shown after shrinking.

By the shrinkability property, for the $\bigtriangledown$-matching problem, it is enough to consider the smallest down-triangle for every pair of points (p,q) from P. Thus, $G_{\bigtriangledown}(P)$ is equivalent to the graph whose vertex set is P and that has an edge between any two vertices p and q if and only if $\bigtriangledown pq$ contains no other points from P. Notice that if $\bigtriangledown pq$ has p as one of its vertices, then $q \in \overline{C_1}(p) \cup \overline{C_2}(p) \cup \overline{C_3}(p)$. The following two properties are simple, but useful.

Property 1. *Let p and p' be two points in the plane. Let $i \in \{1,2,3\}$. The point p is in the cone $C_i(p')$ if and only if the point p' is in the cone $\overline{C}_i(p)$. Moreover, if p is in the cone $C_i(p')$, then $c_i(p',p) = \overline{c_i}(p,p')$.*

PROOF. The first part of the claim is obvious. Now, without loss of generality, assume that $i = 1$ and $p \in C_1(p')$. (See Figure 3). Since $\ell_{\overline{C_1}(p)}$ is the bisector of $\overline{C_1}(p)$ and $\ell_{C_1(p')}$ is the bisector of $C_1(p')$, $\ell_{\overline{C_1}(p)}$ and $\ell_{C_1(p')}$ are parallel lines. Hence, $\overline{c_1}(p, p')$ is the perpendicular distance of p' to the line ℓ_1, which makes an angle 120° with the horizontal and passes though p. Similarly, $c_1(p', p)$ is the perpendicular distance of p to the line ℓ_2, which makes an angle 120° with the horizontal and passes though p'. Hence both $\overline{c_1}(p, p')$ and $c_1(p', p)$ are equal to the perpendicular distance between the lines ℓ_1 and ℓ_2. □

Property 2. *Let P be a point set, $p \in P$ and $i \in \{1,2,3\}$. If $\overline{V}_i(p)$ is non-empty, then, in $G_\triangledown(P)$, the vertex p' corresponding to the point in $\overline{V}_i(p)$ with the minimum value of $\overline{c_i}(p, p')$ is the unique neighbour of vertex p in $\overline{V}_i(p)$.*

PROOF. Assume $\overline{V}_i(p) \neq \emptyset$. For any point p' in $\overline{V}_i(p)$, it is easy to see that $\triangledown pp'$ contains no points outside the cone $\overline{C_i}(p)$. Let p' be the point with the minimum value of $\overline{c_i}(p, p')$. The minimality ensures that $\triangledown pp'$ does not contain any other point other than p and p' from P. Therefore, p and p' are neighbours in $G_\triangledown(P)$.

In order to prove uniqueness, consider any point q in $P \cap \overline{V}_i(p)$ other than p and p'. It can be seen that $\triangledown pq$ contains the point p' and therefore, p and q are not adjacent in $G_\triangledown(P)$. Thus p' is the only neighbour of p in $\overline{V}_i(p)$. □

Consider a point set P and let $p, q \in P$ be two distinct points. By Property 1, $\exists i \in \{1,2,3\}$ such that $p \in \overline{C_i}(q)$ or $q \in \overline{C_i}(p)$; by the general position assumption, both conditions cannot hold simultaneously. Since $\triangledown pq$ has either p or q as a vertex, Property 2 implies that we can construct $G_\triangledown(P)$ as follows. For every point $p \in P$, and for each of the three cones, $\overline{C_i}$, for $i \in \{1,2,3\}$, add an edge from p to the point p' in $\overline{V_i}(p)$ with the minimum value of $\overline{c_i}(p, p')$, if $\overline{V_i}(p) \neq \emptyset$. This definition of $G_\triangledown(P)$ is the same as the definition of the half-Θ_6-graph on negative cones ($\overline{C_i}$), given by Bonichon et al. [1]. We can similarly define the graph $G_\triangledown(P)$ using the cones C_i instead of $\overline{C_i}$, for $i \in \{1,2,3\}$, and show that it is equivalent to the half-Θ_6 graph on positive cones (C_i), given by Bonichon et al. [1]. In Bonichon et al. [1], it was shown that for point sets in general position, the half-Θ_6-graph, the *triangular distance-Delaunay graph* (TD-Del) [3], which are 2-spanners, and the *geodesic embedding* of P, are all equivalent.

The Θ_k-graphs discovered by Clarkson [7] and Keil [8] in the late 80's, are also used as spanners [9]. In these graphs, adjacency is defined as follows: the space around each point p is decomposed into $k \geqslant 2$ regular cones, each with apex p, and a point q of a given cone C is linked to p if, from p, the orthogonal projection of q onto C's bisector [1] is the nearest point in C. In Bonichon et al. [1], it was shown that every Θ_6-graph is the union of two half-Θ_6-graphs, defined by C_i and $\overline{C_i}$ cones. In our notation this is same as the graph $G_\triangledown(P) \cup G_\triangle(P)$, which by definition, is equivalent to $G_{\text{✡}}(P)$. Thus, for a point set in general position, $\Theta_6(P) = G_{\text{✡}}(P)$.

4. Some properties of $G_\triangledown(P)$

4.1. Planarity

Chew defined [3] TD-Delaunay graph to be a planar graph and its equivalence with $G_\triangledown(P)$ graph implies that $G_\triangledown(P)$ is planar. This also follows from the general result that Delaunay graph of any convex distance function is a planar graph [10]. For the sake of completeness, we include a direct proof here.

Lemma 1. *For a point set P, its $G_\triangledown(P)$ is a plane graph, where its edges are straight line segments between the corresponding end-points.*

PROOF. Whenever there is an edge between p and q in $G_\triangledown(P)$, we draw it as a straight line segment from p to q. Notice that this segment always lies within $\triangledown pq$. We will show that this gives a planar embedding of $G_\triangledown(P)$. Consider two edges pq and $p'q'$ of $G_\triangledown(P)$. If the interiors of $\triangledown pq$ and $\triangledown p'q'$ have no point in common, the line segments pq and $p'q'$ can not cross each other. Suppose the interiors of $\triangledown pq$ and $\triangledown p'q'$ share some common area. The case that $\triangledown pq \subseteq \triangledown p'q'$ (or vice versa) is not possible, because in this case $\triangledown p'q'$ contains p and q (or $\triangledown pq$ contains p' and q'), which contradicts its emptiness. Since $\triangledown pq$ and $\triangledown p'q'$ have parallel sides, this implies that one corner of $\triangledown pq$ infiltrates into $\triangledown p'q'$ or vice versa (see Figure 4). Thus their boundaries cross at two distinct points, a and b. Since $P \cap \triangledown p'q' \cap \triangledown p'q' = \emptyset$, the points p and q must be on that portion of the boundary of $\triangledown pq$ that does not lie inside $\triangledown p'q'$. So the line through ab separates pq from $p'q'$. □

[1] Sometimes the definition of Θ_k-graphs allows the orthogonal projection to be made to any ray in the cone C. But in our definition, we stick to the convention that the orthogonal projection is made to the bisector of C.

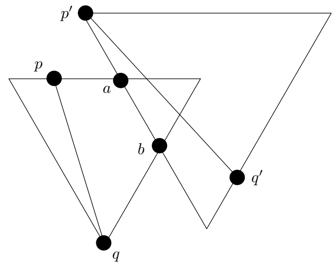


Figure 4: Intersection of $\triangledown pq$ and $\triangledown p'q'$ does not lead to crossing of edges pq and $p'q'$.

Throughout this paper, we use $G_{\triangledown}(P)$ to represent both the abstract graph and its planar embedding described in Lemma 1. The meaning will be clear from the context.

4.2. Connectivity

In this section, we prove that for a point set P, its $G_{\triangledown}(P)$ is connected. As stated in the following lemma, between every pair of vertices, there exist a path with a special structure.

Lemma 2. *Let P be a point set with $p,q \in P$. Then, in $G_{\triangledown}(P)$, there is a path between p and q which lies fully in $\triangledown pq$ and hence $G_{\triangledown}(P)$ is connected.*

PROOF. We will prove this using induction on the rank of the area of $\triangledown pq$. For any pair of distinct points $p,q \in P$, if the interior of $\triangledown pq$ does not contain any point from P, by definition, there is an edge from p to q in $G_{\triangledown}(P)$. By induction, assume that for pairs of points $x,y \in P$ such that the area of $\triangledown xy$ is less than the area of $\triangledown pq$, in the graph in $G_{\triangledown}(P)$, there is a path which lies fully in $\triangledown xy$ between x and y.

If the interior of $\triangledown pq$ does not contain any point from P, there is an edge from p to q in $G_{\triangledown}(P)$. Otherwise, there is a point $x \in P$ which is in the interior of $\triangledown pq$. This implies $\triangledown px \subset \triangledown pq$ and $\triangledown xq \subset \triangledown pq$. Since the area of $\triangledown px$ and the area of $\triangledown xq$ are both less than the area of $\triangledown pq$, by the induction hypothesis, there is a path that lies in $\triangledown px$ between p and x and there is a path that lies in $\triangledown xq$ between x and q. By concatenating these two paths, we get a path which lies in $\triangledown pq$ between p and q. □

4.3. Number of degree-one vertices

In this section, we prove for a point set P, its $G_{\triangledown}(P)$ has at most three vertices of degree one. This fact is important for our proof of the lower bound of the cardinality of a maximum matching in $G_{\triangledown}(P)$.

Definition 1. Let x be a degree-one vertex in $G_{\triangledown}(P)$ and let p be the unique neighbor of x. We say that x uses the horizontal line, if x is below the horizontal line passing through p and points in $P \setminus \{p,x\}$ are all above the horizontal line passing through p. We say that x uses the $120°$ line, if x lies to the right of the $120°$ line passing through p and all points in $P \setminus \{p,x\}$ lie to the left of this line. We say that x uses the $60°$ line, if x lies to the left of the $60°$ line passing through p and all points in $P \setminus \{p,x\}$ lie to the right of this line.

Property 3. *Let x be a degree-one vertex in $G_{\triangledown}(P)$ and let p be the unique neighbor of x such that $x \in V_i(p)$ for $i \in \{1,2,3\}$.*

- *If $x \in V_1(p)$, then x uses the $120°$ line.*
- *If $x \in V_2(p)$, then x uses the $60°$ line.*
- *If $x \in V_3(p)$, then x uses the horizontal line.*

PROOF. To get a pictorial understaning of the property, the reader may refer to Figure 5. Let us consider the case when $x \in V_1(p)$. It is clear that x lies to the right of the 120° line passing through p. Consider a point $y \in P \setminus \{p,x\}$. By the general position assumption, y cannot lie on the 120° line passing through p. If y lies to the right of the 120° line passing through p, since x is already to the right side of the 120° line passing through p, the triangle $\bigtriangledown xy$ will be lying completely to the right side of the 120° line passing through p and therefore $p \notin \bigtriangledown xy$. Hence, by Lemma 2, in $G_{\bigtriangledown}(P)$ there is a path between x and y, which does not pass through p. This contradicts our assumption that p was the unique neighbor of x. Therefore, any point $y \in P \setminus \{p,x\}$ should lie to the left of the 120° line passing through p. Hence, x uses the 120° line.

When $x \in V_2(p)$ or $x \in V_3(p)$, the proofs are similar. □

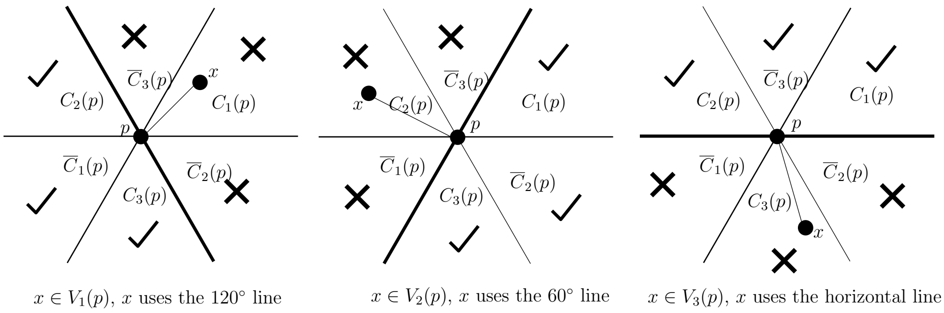


Figure 5: Illustration of Property 3. The cones around p which are allowed to have points from $P \setminus \{p,x\}$ are marked with ✓and the other cones around p are marked with ×.

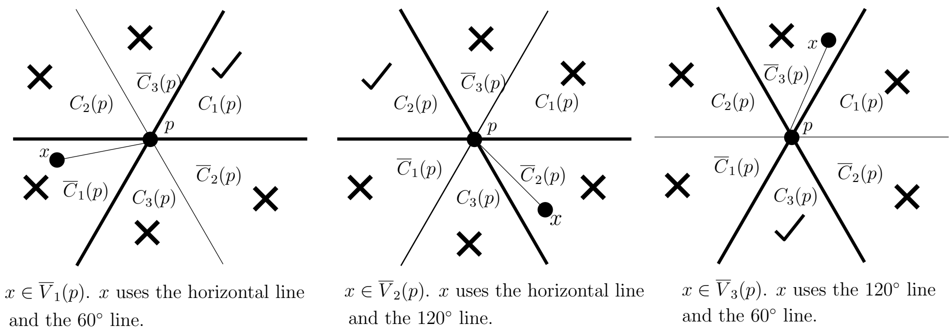


Figure 6: Illustration of Property 4. The cones around p which are allowed to have points from $P \setminus \{p,x\}$ are marked with ✓and the other cones around p are marked with ×.

Property 4. *Let x be a degree-one vertex in $G_{\bigtriangledown}(P)$ and let p be the unique neighbor of x such that $x \in \overline{V}_i(p)$ for $i \in \{1,2,3\}$.*

- *If $x \in \overline{V}_1(p)$, then x uses the horizontal line and the 60° line.*
- *If $x \in \overline{V}_2(p)$, then x uses the horizontal line and the 120° line.*
- *If $x \in \overline{V}_3(p)$, then x uses the 60° line and the 120° line.*

PROOF. To get a pictorial understaning of this property, the reader may refer to Figure 6. This property can be proved using similar arguments as in the proof of Property 3. We omit the proof here, to avoid redundancy. □

Property 5. *Let x be a degree-one vertex in $G_{\bigtriangledown}(P)$ and p be the unique neighbor of x. Let $x' \in P \setminus \{x\}$ be another degree-one vertex in $G_{\bigtriangledown}(P)$.*

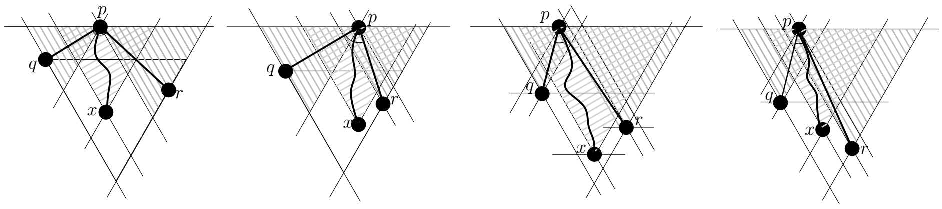


Figure 7: Case 1. $q \in \overline{C_1}(p)$ and $r \in \overline{C_2}(p)$, Case 2. $q \in \overline{C_1}(p)$ and $r \in C_3(p)$, Case 3. $r \in \overline{C_2}(p)$ and $q \in C_3(p)$, Case 4. $q, r \in C_3(p)$.

- *If x uses the horizontal line, then, x' cannot use the horizontal line.*
- *If x uses the 60° line, then, x' cannot use the 60° line.*
- *If x uses the 120° line, then, x' cannot use the 120° line.*

PROOF. We prove only the first part. Proofs of the other parts are similar.

Suppose x uses the horizontal line. By definition, x lies below the horizontal line passing through p and $x' \in P \setminus \{x\}$ lies on or above above this line. This implies that x lies below the horizontal line through x'. If x' also uses the horizontal line, since $x \in P \setminus \{x'\}$, by a symmetric argument, we can show that x' lies below the horizontal line through x. Since these two conditions are not simultaneously possible, we can conclude that if x uses the horizontal line, then x' cannot use the horizontal line. □

Lemma 3. *For a point set P, its $G_{\triangledown}(P)$ has at most three vertices of degree one.*

PROOF. For contradiction, assume that there are four degree-one vertices x_1, x_2, x_3 and x_4 in $G_{\triangledown}(P)$. From Property 3 and Property 4, we can see that each x_i uses at least one of the three types of reference lines: either the horizontal line, or the 60° line or the 120° line. By pigeonhole principle, at least two among these four degree-one vertices use the same type of reference line.

Without loss of generality, assume that x_1 and x_2 uses the same type of reference line. If x_1 and x_2 are adjacent to each other, these two degree-one vertices will form a connected component in $G_{\triangledown}(P)$, which will contradict the fact that $G_{\triangledown}(P)$ is connected. Therefore, x_1 and x_2 are non-adjacent. Hence, by Property 5, x_1 and x_2 cannot use the same type of reference line.

Therefore, we can conclude that $G_{\triangledown}(P)$ has at most three vertices of degree one. □

4.4. Internal triangulation

If all the internal faces of a plane graph are triangles, we call it an internally triangulated plane graph. In this section, we will prove that for a point set P, the plane graph $G_{\triangledown}(P)$ is internally triangulated. This property will be used in Section 5 to derive the lower bound for the cardinality of maximum matchings in $G_{\triangledown}(P)$.

Lemma 4. *For a point set P, all the internal faces of $G_{\triangledown}(P)$ are triangles.*

PROOF. Consider an internal face f of $G_{\triangledown}(P)$. We need to show that f is a triangle. Let p be the vertex with the highest y-coordinate among the vertices on the boundary of f. Since f is an internal face, p has at least two neighbours on the boundary of f. Let q and r be the neighbours of p on the boundary of f such that r is to the right of the line passing through q and making an angle of 120° with the horizontal and any other neighbour of p on the boundary of f is to the right of the line passing through r and making an angle 120° with the horizontal. Because of the general position assumption, q and r can be uniquely determined.

We will prove that qr is also an edge on the boundary of f and there is no point from P in the interior of the triangle whose vertices are p, q and r. This will imply that the face f is the triangle whose vertices are p, q and r.

We know that $q, r \in \overline{C_1}(p) \cup \overline{C_2}(p) \cup C_3(p)$. By Property 2, it cannot happen that both $q, r \in \overline{C_i}(p)$, for any $i \in \{1, 2\}$. Other possibilities are shown in Figure 7, where q is assumed to be above r. An analogous argument can be made when r is above q as well. Since pq and pr are edges in $G_{\triangledown}(P)$, we know that $\triangledown pq \cap (P \setminus \{p, q\}) = \emptyset$ and $\triangledown pr \cap (P \setminus \{p, r\}) = \emptyset$.

Notice that, the area bounded by the lines (1) the horizontal line passing through p, (2) the line passing through q and making an angle of 120° with the horizontal, and (3) the line passing through r and making an angle of 60° with the horizontal, will define an equilateral down triangle with p, q and r on its boundary. Let us denote this triangle by $\triangledown pqr$.

Claim 1. $\triangledown pqr \cap (P \setminus \{p,q,r\}) = \emptyset$.

PROOF. For contradiction, let us assume that there exists a point $x \in \triangledown pqr \cap (P \setminus \{p,q,r\})$. Because of the general position assumption, x cannot be on the boundary of $\triangledown pqr$. Therefore, $\triangledown px$ does not contain q and r. By Lemma 2, in $G_\triangledown(P)$, there exists a path between p and x which lies inside $\triangledown px$. Let this path be $X = v_1 v_2, \ldots, v_k = x$. Since $\triangledown pq \cap P \setminus \{p,q\} = \emptyset$, $\triangledown pr \cap P \setminus \{p,r\} = \emptyset$ and $q, r \notin \triangledown px$, we know that all vertices in the path $X = v_1 v_2, \ldots, v_k = x$ lie inside the region $R = (\triangledown px \setminus (\triangledown pq \cup \triangledown pr)) \cup \{p\}$.

Let C be the cone with apex p bounded by the rays pq and pr. Observe that for any point $v \in R$, the line segment pv lies inside the cone C. Since $v_2 \in R$ and pv_2 is an edge (in the path from p to x), the line segment corresponding to the edge pv_2 lies inside C in $G_\triangledown(P)$.

If the point v_2 is outside the face f, edge pv_2 will cross the boundary of f, which is contradicting the planarity of $G_\triangledown(P)$. Since v_2 cannot be outside the face f, the edge pv_2 belongs to the boundary of f. Since v_2 lies inside the cone C and $v_2 \in R$, this means that v_2 is a neighbour of p on the boundary of f such that v_2 is to the left of the the line passing through r and making an angle of 120° with the horizontal. This is a contradiction to our assumption that q is the only neighbour of p on the boundary of f, lying to the left of the the line passing through r and making an angle of 120° with the horizontal. □

Let us continue with the proof of Lemma 4. Since the triangle with vertices p, q and r is inside the triangle $\triangledown pqr$, from the above claim, it is clear that there is no point from P, other than the points p, q and r, inside the triangle whose vertices are p, q and r. Since the edges pq and pr belong to the boundary of f, to show that f is a triangle, it is now enough to prove that qr is also an edge in $G_\triangledown(P)$. This fact also follows from the above claim as explained below.

Since $\triangledown qr \subseteq \triangledown pqr$, by the claim above, $\triangledown qr$ cannot contain any point from P other than p, q and r. Moreover, since p lies above q and r, we know that $p \notin \triangledown qr$. Therefore, $\triangledown qr \cap (P \setminus \{q,r\}) = \emptyset$. Therefore, qr is an edge in $G_\triangledown(P)$.

Thus, f has to be a triangle bounded by the edges pq, qr and pr. □

Corollary 1. *For a point set P, all the cut vertices of $G_\triangledown(P)$ lie on its outer face.*

PROOF. Consider any vertex v of $G_\triangledown(P)$ which is not on its outer face. Since $G_\triangledown(P)$ is internally triangulated, each neighbour of v in $G_\triangledown(P)$ lies on a cycle in the graph $G_\triangledown(P) \setminus v$. Since $G_\triangledown(P)$ is connected, $G_\triangledown(P) \setminus v$ remains connected. Thus, v cannot be a cut vertex. □

Combining Lemma 1, Lemma 2, Lemma 3 and Lemma 4, we get:

Theorem 1. *For a point set P, $G_\triangledown(P)$ is a connected and internally triangulated plane graph, having at most three degree-one vertices.*

5. Maximum matching in $G_\triangledown(P)$

In this section, we show that for any point set P of n points, $G_\triangledown(P)$ contains a matching of size $\lceil \frac{n-1}{3} \rceil$; i.e, at least $2\left(\lceil \frac{n-1}{3} \rceil\right)$ vertices are matched. In order to do this, we will prove the following general statement:

Lemma 5. *Let G be a connected and internally triangulated plane graph, having at most three vertices of degree one. Then, G contains a matching of size at least $\left\lceil \frac{|V(G)|-1}{3} \right\rceil$.*

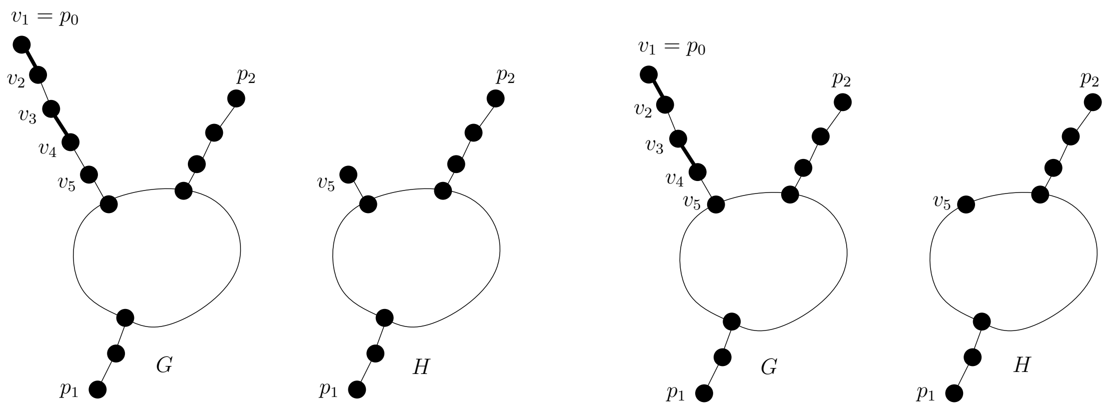
An overview of the proof. Let G be a graph on n vertices, satisfying the assumptions of Lemma 5. Since G is a connected graph, the lemma holds trivially when $n \leq 4$. Therefore, we assume that $n \geq 5$. We construct an auxiliary graph G' such that it is a 2-connected planar graph of minimum degree at least 3, and then make use of the following theorem of Nishizeki [11] to get a lower bound on the size of a maximum matching of G'.

Theorem 2 ([11]). *Let G' be a connected planar graph with n' vertices having minimum degree at least* 3 *and let M' be a maximum matching in G'. Then,*

$$|M'| \geq \begin{cases} \lceil \frac{n'+2}{3} \rceil & \text{when } n' \geq 10 \text{ and } G' \text{ is not 2-connected} \\ \lceil \frac{n'+4}{3} \rceil & \text{when } n' \geq 14 \text{ and } G' \text{ is 2-connected} \\ \lfloor \frac{n'}{2} \rfloor & \text{otherwise} \end{cases}$$

Using the above result, we will derive a lower bound on the size of a maximum matching of G.

Pre-processing. Let the degree-one vertices of G be denoted by $p_0, p_1, \dots, p_{k-1}$. By our assumption, $k \leq 3$. If $k = 3$, and for each $0 \leq i \leq 2$ the unique neighbor of p_i is a degree two vertex in G, we do some pre-processing to convert it into a graph in which this condition does not hold. To understand this pre-processing easily, the reader may refer to Figure 8. Let $\mathscr{P}$ be the path $(p_0 = v_1, v_2, \dots, v_{2t})$ of maximum length in G such that $\mathscr{P}$ contains an even number of vertices and $v_2, \dots, v_{2t}$ are of degree two in G. We have $t \geq 1$. Let v_{2t+1} be the neighbor of v_{2t}, other than v_{2t-1} in G. Let H be the plane graph obtained from the plane graph G, by deleting the vertices $v_1, v_2, \dots, v_{2t}$, along with their incident edges. It is clear that $\mathscr{P}$ has a unique maximum matching of size t and a maximum matching of G can be obtained by taking the union of a maximum matching in H and the maximum matching in $\mathscr{P}$.



(a) (b)

Figure 8: Pre-processing step constructing H from G. In both the cases above, the path $\mathscr{P}$=$(v_1, v_2, \dots, v_4)$. The union of a maximum matching in H and the matching $\{(v_1, v_2), v_3, v_4)\}$ in $\mathscr{P}$ gives a maximum matching of G. (a) In G, the vertex v_5 is of degree two. It becomes a degree-one vertex in H and its neighbor has degree at least three in H. (b) In G, the vertex v_5 has degree greater than two. H has only two vertices of degree one.

Since $k = 3$ and G is connected, it is easy to see that the vertex v_{2t+1} is not a degree-one vertex in G. Since the degree of v_{2t+1} in H is one less than its degree in G, the degree of v_{2t+1} is at least one in H. By the maximality of $\mathscr{P}$, we can conclude that one of the following is true. If v_{2t+1} is a degree-one vertex in H, then, the unique neighbor of v_{2t+1} has degree at least 3 in H (as in Figure 8(a)). If v_{2t+1} has degree greater than one in H, then, H has at most two degree-one vertices, p_1 and p_2 (as in Figure 8(b)).

The properties of the path $\mathscr{P}$ ensures that H is connected. Since all the removed vertices $v_1, \dots, v_{2t}$ were of degree less than three, they were all on the outer face of the internally triangulated graph G. Therefore, H remains internally triangulated as well.

When at least one of the degree-one vertices of G has a neighbor of degree greater than two or when $k \leq 2$ we initialize $H = G$.

From the construction of H, we can make the following observation.

Property 6. *H is a connected and internally triangulated plane graph. H has at most three degree-one vertices. If H has three degree-one vertices, then, one of the degree-one vertices has a neighbor of degree at least three. If M_H is a maximum matching in H, then, G has a matching of size $|M_H|+t$, where t is an integer given by $\frac{|V(G)|-|V(H)|}{2}$.*

Construction of the auxiliary graph G'. Now we describe the construction of a supergraph G' of H such that G' will satisfy the assumptions of Theorem 2; i.e. we want G' to be a bi-connected planar graph of minimum degree at least 3. Our construction will also ensure that there exist either a single vertex v or two vertices u and v in G', such that every edge in $E(G')\setminus E(H)$ has one of its end points at u or v. Since a matching M' of G' can have at most one edge incident at each of u and v, this implies that H has a matching of size at least $M'-2$.

We initialize G' to be the same as H. Let the degree-one vertices of H be denoted by $q_0,q_1,\ldots,q_{h-1}$. If H has no degree-one vertices, we consider h to be zero. By Property 6, we have $h\leq 3$. If $h=0$ or 1, the modification of G' is simple. We insert a new vertex x in the outer face of G' and add edges between x and all other vertices which were already on the outer face of G' (i.e, add edges between the new vertex x and vertices which were on the outer face of H). This transformation maintains planarity. All vertices in G' except the vertex q_0 (present only when $h=1$) have degree at least three now. If $h=1$, the degree of q_0 has become two in G' at this stage. In this case, let f be a face of the current graph G', containing both q_0 and x. Modify G' by inserting a new vertex y inside f and adding edges from this new vertex to all other vertices belonging to f. As earlier, this transformation maintains planarity. Now, the degree of q_0 becomes 3 and thus G' achieves minimum degree 3. Notice that, when $h=0$ every edge in $E(G')\setminus E(H)$ is incident at x and when $h=1$ every edge in $E(G')\setminus E(H)$ is incident at x or y.

If $h=2$ or $h=3$, consider a simple closed curve $\mathscr{C}$ in the plane such that (1) the entire graph H (all its vertices and edges) lies inside the bounded region enclosed by $\mathscr{C}$, (2) the vertices of H which lie on $\mathscr{C}$ are precisely the degree-one vertices of H, (3) except for the end points, every edge of H lies in the interior of the bounded region enclosed by $\mathscr{C}$. The region of the outer face of H, bounded by the curve $\mathscr{C}$, can be divided into h regions $R_0,\ldots,R_{h-1}$, where R_i is the region bounded by the edge at q_i, the edge at $q_{(i+1)\mod h}$ and the boundary of the outer face of H and the curve $\mathscr{C}$. (Here onwards, in this subsection we assume that indices of vertices and regions are taken modulo h). Notice that every vertex on the outer-face of H lies on at least one of these regions and q_i lies on the regions R_i and R_{i-1}, for $0\leq i\leq h-1$.

When $h=2$, we insert two new vertices x,y into G'. (See Figure 9(a)). Three types of new edges are added in G': (1) between x and y (2) between the vertex x and all the vertices of H which lie on the region R_0 and (3) between y and all the vertices of H which lie on the region R_1. This transformation maintains planarity. (We can imagine x and y to be points on the boundary of the regions R_0 and R_1 respectively, but distinct from any point on the boundary of the outer face of H. Edges between the new vertex x and old vertices on R_0 can be drawn inside R_0 and edges between y and the old vertices on R_1 can be drawn inside R_1. The edges among the new vertices x and y can be drawn outside these regions, except at their end points). Both of the vertices q_0 and q_1 lie in both the regions R_0 and R_1. Therefore, q_0 and q_1 becomes adjacent to both x and y in G' and hence degrees of vertices q_0, q_1, x, y are all at least 3 in G'. Since H was an internally triangulated planar graph, all the degree two vertices of H were on the outer face of H. Therefore, each of them gets at least one new neighbor (x or y) in G'. Therefore, minimum degree of G' is at least 3. In this case also, every edge in $E(G')\setminus E(H)$ is incident at x or y. When $h=3$, Property 6 ensures that the neighbor of one of the degree-one vertices of H has degree at least 3. Without loss of generality, assume that the neighbor of q_0 has degree at least 3 in H. In this case, we insert one new vertex x into G'. (See Figure 9(b)). Three types of new edges are added in G': (1) between x and q_0 (2) between q_0 and all the other vertices of H which were on the regions R_0 and R_2 (3) between x and all the vertices of H which were on the region R_1. This transformation also maintains planarity. (We can imagine x to be a point on the boundary of the region R_1, but distinct from any point on the boundary of the outer face of H. Edges between q_0 and the other vertices on R_0 can be drawn inside R_0 and edges between q_0 and the other vertices on R_2 can be drawn inside R_2. Edges between x and the other vertices on R_1 can be drawn inside R_1. The edges among the new vertices x and q_0 can be drawn outside these regions, except at their end points). Vertices q_1 and q_2 become adjacent to both q_0 and x in G'. Therefore, degrees of q_0, q_1, q_2 are at least 3. In addition, q_0 is also adjacent to x. Therefore, degree of x is also at least three in G'. Suppose vertex v was the (unique) neighbor of q_0 in H. By Property 6, v has degree at least three in H and hence also in G'. All degree two vertices of H, which belonged to R_0 or R_2 were non-adjacent to q_0 in H; but are adjacent to q_0 in G'. Thus, they attain degree at least 3 in G'. All degree two vertices of H, which belonged to R_2 gets a new neighbor x in G' and attain degree three. Thus, the minimum degree of G' is at least 3 in this case as well. Every edge in $E(G')\setminus E(H)$ is incident at x or q_0.

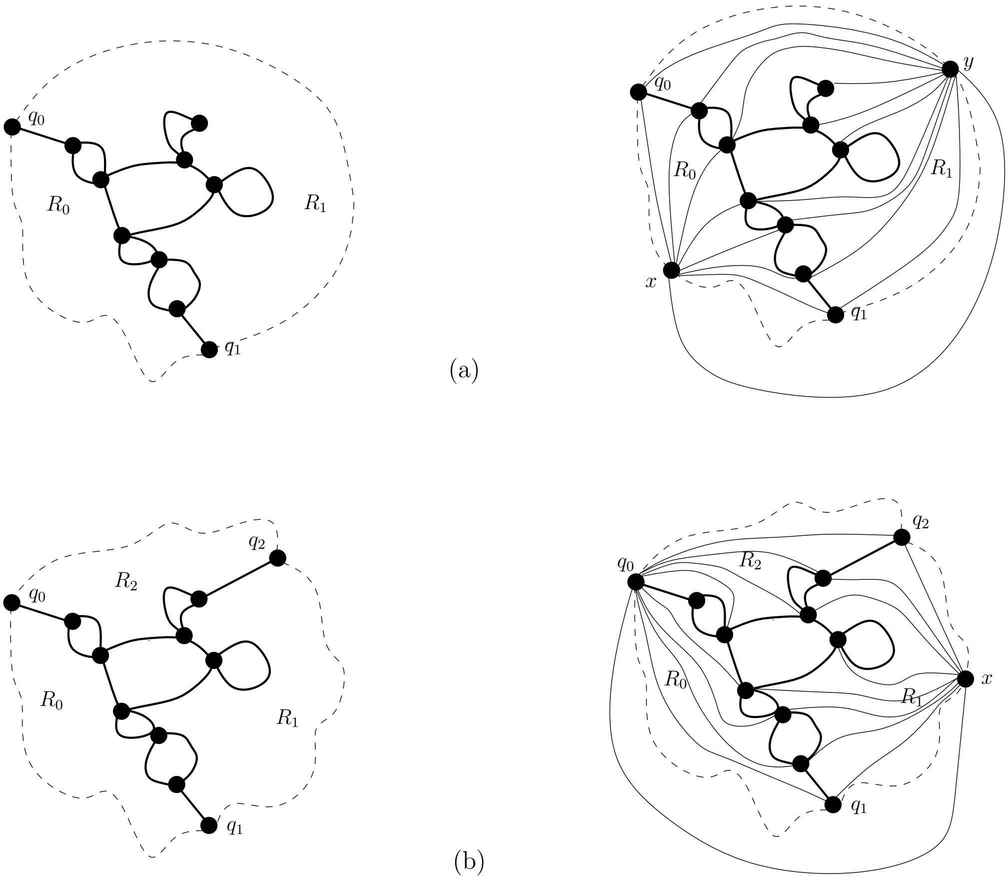


Figure 9: (a) Modification done when H has two degree-one vertices. Every edge in $E(G') \setminus E(H)$ is incident at x or y. (b) Modification done when H has three degree-one vertices. Every edge in $E(G') \setminus E(H)$ is incident at q_0 or x.

From the description above, we can make the following observation.

Property 7. *G' is a planar graph of minimum degree at least three, with $|V(H)|+1 \leq |V(G')| \leq |V(H)|+2$. There exist either a single vertex u or two vertices u and v in G', such that every edge in $E(G') \setminus E(H)$ has one of its end points at u or v.*

Claim 2. *The graph G' is 2-connected.*

PROOF. In all the different cases above, it is easy to observe that none of the newly inserted vertices can be a cut vertex of G'.

Consider an arbitrary vertex $v \in V(H)$. If v is not a cut vertex of H, then, $H \setminus v$ is connected. Since G' has minimum degree at least 3, any newly added vertex has a neighbor in $V(H) \setminus \{v\}$ in the graph G'. Therefore, $G' \setminus v$ remains connected. Therefore, none of the non-cut vertices of H can be a cut vertex of G'. In particular, none of the degree-one vertices of H can be a cut vertex of G'.

If v is a cut vertex in H, v was on the outer face of H, because H was internally triangulated. It is clear that if two vertices $v_1, v_2 \in V(H)$ are in the same connected component of $H \setminus v$, they are in the same connected component of $G' \setminus v$ as well. If C_1 and C_2 are two components of $H \setminus v$, then we know that there are vertices $v_1 \in V(C_1)$ and $v_2 \in V(C_2)$, such that v_1 and v_2 are neighbors of v on the outer face of H.

When $h \leq 2$, vertices v_1 and v_2 have an edge to at least one of the newly inserted vertices in G'. Since the induced subgraph of G' on the newly inserted vertices is connected, in G' we get a path from v_1 to v_2 in which all the intermediate vertices are newly inserted vertices in G'. When $h = 3$, we have two cases to consider. It is possible that

v_1 or v_2 is same as the vertex q_0 itself. If this is not the case, v_1 and v_2 have edges to either q_0 or the new vertex x in G'. In either case, since there is an edge between q_0 and x in G', we get a path from v_1 to v_2 in $G' \setminus v$. Thus, in all cases when $h \geq 3$, any two components C_1 and C_2 of $H \setminus v$ become part of the same connected component of $G' \setminus v$. Moreover, by the construction of G', the degree-one vertices of H and the vertices in $V(G') \setminus V(H)$ are part of the same component of $G' \setminus v$. This implies that $G' \setminus v$ has only a single connected component and hence, v is not a cut vertex of G'.

Thus, G' is 2-connected. □

A lower bound for the cardinality of a maximum matching in G. By Property 7 and Claim 2, the auxiliary graph G' is a 2-connected planar graph of minimum degree at least 3. Let $n' = |V(H)| + t_1$ be the number of vertices of G', where $t_1 = 1$ or $t_1 = 2$ by Property 7. By Theorem 2, the cardinality of a maximum matching M' in G' is at least $\left\lceil \frac{n'+4}{3} \right\rceil$ when $n' \geq 14$ and $|M'| \geq \lfloor \frac{n'}{2} \rfloor$, otherwise. Since H is a subgraph of G', if we delete the edges in M' which belong to $E(G') \setminus E(H)$, we get a matching M_H of H. Since M' is a matching in G', M' can have at most one edge incident at any vertex of G'. Hence, by Property 7, there can be at most two edges in $M' \cap (E(G') \setminus E(H))$. Therefore, we have $|M_H| \geq |M'| - 2$. From this, we get,

$$|M_H| \geq \begin{cases} \left\lceil \frac{|V(H)|+t_1+4}{3} \right\rceil - 2, & \text{when } |V(H)| + t_1 \geq 14 \\ \left\lfloor \frac{|V(H)|+t_1}{2} \right\rfloor - 2, & \text{otherwise} \end{cases}$$

By Property 6, G has a matching M of size $|M_H| + t$, where t is an integer, given by $\frac{|V(G)|-|V(H)|}{2}$. By substituting the lower bound for $|M_H|$, we get,

$$|M| \geq \begin{cases} \left\lceil \frac{|V(H)|+t_1+4}{3} \right\rceil - 2 + t, & \text{when } |V(H)| + t_1 \geq 14 \\ \left\lfloor \frac{|V(H)|+t_1}{2} \right\rfloor - 2 + t, & \text{otherwise} \end{cases}$$

Since $t_1 = 1$ or 2 and $t = |V(G)| - |V(H)| \geq 0$, this gives

$$|M| \geq \begin{cases} \left\lceil \frac{|V(G)|-1}{3} \right\rceil, & \text{when } |V(H)| \geq 13 \\ \left\lfloor \frac{|V(G)|-3}{2} \right\rfloor, & \text{otherwise} \end{cases}$$

Whenever $|V(G)| \geq 7$, from the above inequality, we get $|M| \geq \left\lceil \frac{|V(G)|-1}{3} \right\rceil \geq 2$. Since G has at most three vertices of degree one, when $|V(G)| \geq 5$, G cannot be a star with $|V(G)| - 1$ leaves. Therefore, when $|V(G)| \geq 5$, $|M| \geq 2$. When $|V(G)| > 1$, since G is connected, we get $|M| \geq 1$. From this discussion, we can conclude that, in all cases, $|M| \geq \left\lceil \frac{|V(G)|-1}{3} \right\rceil$. This concludes the proof of Lemma 5.

As an immediate corollary of Lemma 5 and Theorem 1, we get:

Theorem 3. *For any point set P of n points in general position, $G_{\triangledown}(P)$ contains a matching of size* $\left\lceil \frac{n-1}{3} \right\rceil$.

Some graphs for which our bound is tight. In Figure 10 (a), a point set P consisting of 15 points and the corresponding graph $G_{\triangledown}(P)$ is given. This graph has a maximum matching (shown in thick lines) of size $\left\lceil \frac{|P|-1}{3} \right\rceil = 5$. This is the same example as given by Panahi et al. [2]. By adding more triplets of points (a_i, b_i, c_i), $i > 4$, into P, following the same pattern, we can show that for any $n \geq 15$ which is a multiple of 3, there is a point set P of n points in general position, such that a maximum matching in $G_{\triangledown}(P)$ is of cardinality $\left\lceil \frac{|P|-1}{3} \right\rceil$. We can also show that, for any $n \geq 13$, which is one more than a multiple of three, there is a point set P' on n points in general position, such that a maximum matching in $G_{\triangledown}(P')$ is of cardinality $\left\lceil \frac{|P'|-1}{3} \right\rceil$. For example, take the point set $P' = P \setminus \{a_0, b_0\}$ where P is the point

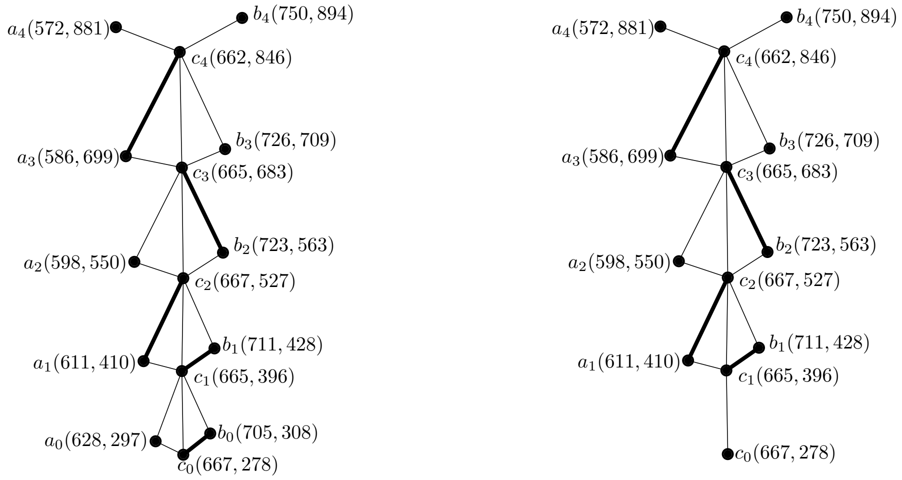


Figure 10: (a) A point set P with 15 points in general position, where $G_{\bigtriangledown}(P)$ has a maximum matching of size $\lceil \frac{n-1}{3} \rceil = 5$ [2]. (b) A point set P with 13 points in general position, where $G_{\bigtriangledown}(P)$ has a maximum matching of size $\lceil \frac{n-1}{3} \rceil = 4$.

set of triplets described in the paragraph above. Figure 10 (b) illustrates this for $n = 13$, in which case a maximum matching in $G_{\bigtriangledown}(P')$ has cardinality $\left\lceil \frac{|P'|-1}{3} \right\rceil = 4$. Similarly, for any $n \geq 14$, which is two more than a multiple of three, there is a point set P' on n points in general position, such that a maximum matching in $G_{\bigtriangledown}(P')$ is of cardinality $\left\lceil \frac{|P'|-1}{3} \right\rceil$. For example, take the point set $P' = P \setminus \{a_0\}$ where P is the point set of triplets described in the paragraph above. From the examples above, it is clear that the bound given in Theorem 3 is tight.

5.1. A 3-connected down triangle graph without perfect matching

The example given by Panahi et al. [2], for a point set P for which $G_{\bigtriangledown}(P)$ has a maximum matching of size $\lceil \frac{n-1}{3} \rceil$, contained many cut vertices. However, for general planar graphs, we get a better lower bound for the size of a maximum matching, when the connectivity of the graph increases. By Theorem 2, we know that any 3-connected planar graph on n vertices has a matching of size $\lceil \frac{n+4}{3} \rceil$, if $n \geq 14$ and has a matching of size $\lfloor \frac{n}{2} \rfloor$ if $n < 14$ or it is 4-connected. Hence, it was interesting to see whether there exist a point set P in general position, with an even number of points, such that $G_{\bigtriangledown}(P)$ is 3-connected but does not contain a perfect matching. The answer is positive. Consider the graph given in Figure 11 (a), which shows a point set P of 18 points in general position and the corresponding graph $G_{\bigtriangledown}(P)$. This graph has a maximum matching (shown in thick lines) of size 8. We can follow the pattern and go on adding points a_i, b_i and c_i, for $i > 4$ to the point set such that when $P = \{a_0, b_0, c_0, \ldots, a_k, b_k, c_k, p_1, p_2, p_3\}$, $G_{\bigtriangledown}(P)$ is a 3-connected graph with a maximum matching of size $\left\lceil \frac{|P|+5}{3} \right\rceil$. It can be verified that $G_{\bigtriangledown}(P \setminus \{a_0\})$ and $G_{\bigtriangledown}(P \setminus \{a_0, b_0\})$ are also 3-connected and their maximum matchings have size $\left\lceil \frac{|P|+5}{3} \right\rceil$. (See Figure 11 (b) for the case when $|P| = 16$). Thus, for 3-connected down triangle graphs corresponding to point sets in general position, the best known lower bound for maximum matching is $\lceil \frac{n+4}{3} \rceil$ and the examples we discussed above show that it is not possible to improve the bound above $\lceil \frac{n+5}{3} \rceil$.

6. Some properties of $G_{✡}(P)$

In this section, we prove that for a point set P, the 2-connectivity structure of $G_{✡}(P)$ is simple and $G_{✡}(P)$ can have at most $5n - 11$ edges.

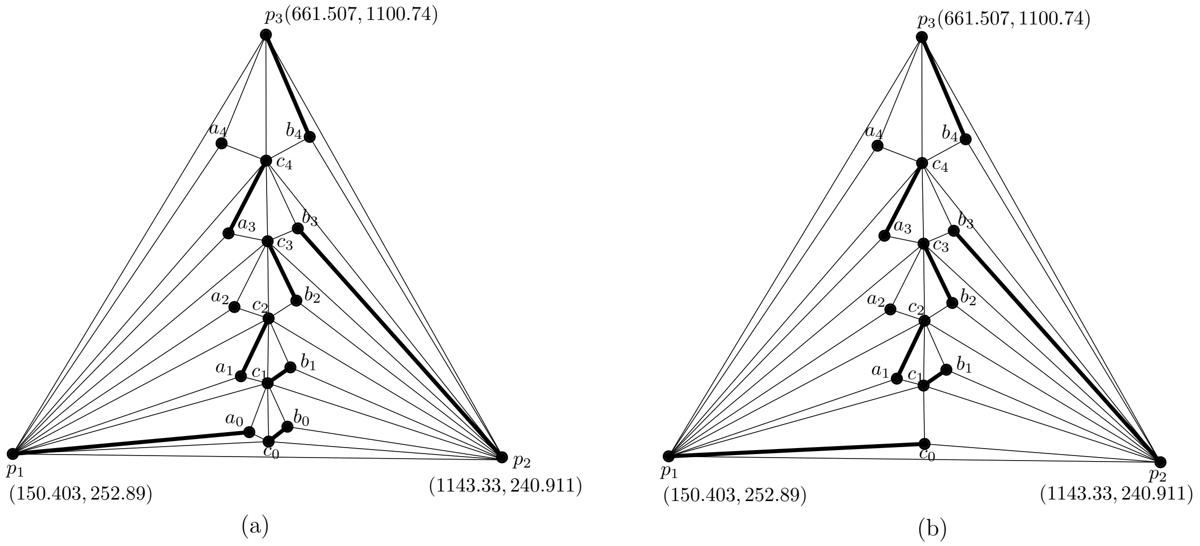


Figure 11: (a) A point set P with 18 points in general position, where $G_{\bigtriangledown}(P)$ is 3-connected and has a maximum matching of size $\lceil \frac{n+5}{3} \rceil$. (b) A point set P with 16 points in general position, where $G_{\bigtriangledown}(P)$ is 3-connected and has a maximum matching of size $\lceil \frac{n+5}{3} \rceil$. The points with their co-ordinates unspecified have the same co-ordinates as in Figure 10.

6.1. Block cut point graph

Let $G(V,E)$ be a graph. A block of G is a maximal connected subgraph having no cut vertex. The block cut point graph of G is a bipartite graph $B(G)$ whose vertices are cut-vertices of G and blocks of G, with a cut-vertex x adjacent to a block X if x is a vertex of block X. The block cut point graph of G gives information about the 2-connectivity structure of G.

Since $G_{\davidsstar}(P)$ is the union of two connected graphs $G_{\bigtriangledown}(P)$ and $G_{\bigtriangleup}(P)$ (Lemma 2), it is connected and hence its block-cut point graph is a tree [12]. We will show that the block cut point graph of $G_{\davidsstar}(P)$ is a simple path. We use the following lemma in our proof.

Lemma 6. *Let P be a point set and $p \in P$ be a cut vertex of $G_{\davidsstar}(P)$. Then, there exists an $i \in \{1,2,3\}$ such that $V_i(p) \neq \emptyset$, $\overline{V_i}(p) \neq \emptyset$ and for all $j \in \{1,2,3\} \setminus \{i\}$, $V_j(p) = \emptyset$ and $\overline{V_j}(p) = \emptyset$. Moreover, $G_{\davidsstar}(P) \setminus p$ has exactly two connected components, one containing all vertices in $V_i(p)$ and the other containing all vertices of $\overline{V_i}(p)$.*

PROOF. Since p is a cut vertex of $G_{\davidsstar}(P)$, we know that there exist $v_1, v_2 \in P$ that are in different components of $G_{\davidsstar}(P) \setminus p$. We will show that v_1 and v_2 should be in opposite cones with reference to the apex point p.

Without loss of generality, assume that $v_1 \in A_1(p) \cap P \setminus \{p\}$. If $v_2 \in (A_1(p) \cup A_2(p) \cup A_6(p)) \cap (P \setminus \{p\})$, then, $p \notin \bigtriangledown v_1 v_2$ and hence by Lemma 2, there is a path in $G_{\bigtriangledown}(P)$ between v_1 and v_2 that does not pass through p, which is not possible. Similarly, if $v_2 \in (A_3(p) \cup A_5(p)) \cap (P \setminus \{p\})$, then, $p \notin \bigtriangleup v_1 v_2$ and there is a path in $G_{\bigtriangleup}(P)$ between v_1 and v_2 that does not pass through p, which is not possible. Therefore, $v_2 \in A_4(p)$, the cone which is opposite to $A_1(p)$ which contains v_1. Thus any two points v_1 and v_2 which are in different connected components of $G_{\davidsstar}(P) \setminus p$, are in opposite cones around p.

Let C_1 and C_2 be two connected components of $G_{\davidsstar}(P) \setminus p$ with $v_1 \in C_1$ and $v_2 \in C_2$. Without loss of generality, assume that such $v_1 \in V_1(p)$ and $v_2 \in \overline{V_1}(p)$. From the paragraph above, we know that every vertex of $G_{\davidsstar}(P) \setminus p$ which is not in C_1 is in $\overline{V_1}(p)$ and every vertex of $G_{\davidsstar}(P) \setminus p$ which is not in C_2 is in $V_1(p)$. This implies that for all $j \in \{2,3\}$, $V_j(p) = \emptyset$ and $\overline{V_j}(p) = \emptyset$. This proves the first part of our lemma.

For any $v_1, v_2 \in \overline{V_i}(p)$, we have $p \notin \bigtriangledown v_1 v_2$ and hence by Lemma 2, there is a path in $G_{\bigtriangledown}(P)$ between v_1 and v_2 that does not pass through p. Similarly, for any $v_1, v_2 \in V_i(p)$, $p \notin \bigtriangleup v_1 v_2$ and there is a path in $G_{\bigtriangleup}(P)$ between v_1 and v_2 that does not pass through p. Therefore, there are exactly two connected components in $G_{\davidsstar}(P) \setminus p$, one containing all vertices in $V_i(p)$ and the other containing all vertices of $\overline{V_i}(p)$. □

Theorem 4. *Let P be a point set in general position and let k be the number of blocks of $G_{\davidsstar}(P)$. Then, the blocks of $G_{\davidsstar}(P)$ can be arranged linearly as $B_1, B_2, \ldots B_k$ such that, for $i > j$, $B_i \cap B_j$ contains a single (cut) vertex p_i when $j = i+1$ and $B_i \cap B_j$ is an empty graph otherwise. That is, the block cut point graph of $G_{\davidsstar}(P)$ is a path.*

PROOF. If $G_{\davidsstar}(P)$ is two-connected, there is only a single block and the lemma is trivially true.

Since $G_{\davidsstar}(P)$ is a connected graph, its block cut point graph is a tree. Any two blocks can have at most one vertex in common and the common vertex is a cut vertex. From Lemma 6, we also know that three or more blocks cannot share a common (cut) vertex. If a block B_i of $G_{\davidsstar}(P)$ is such that, in the block cut point graph of $G_{\davidsstar}(P)$, the node corresponding to block B_i is a leaf node, B_i is adjacent to only one another block and they share a single (cut) vertex.

If the node corresponding to B_i is not a leaf node of the block cut point graph, we know that B_i shares (distinct) common vertices with at least two other blocks $B_{i'}$ and $B_{i''}$. Therefore, two vertices in B_i are cut vertices of $G_{\davidsstar}(P)$. Let v_1, v_2 be these cut vertices. We will show that there cannot be a third such cut vertex in B_i.

By Lemma 6, we know that $G_{\davidsstar}(P) \setminus v_1$ has exactly two components and since B_i is 2-connected initially, all vertices of B_i except v_1 are in the same connected component of $G_{\davidsstar}(P) \setminus v_1$. By Lemma 6, all vertices of B_i lie in the same (designated) cone with apex v_1. Without loss of generality, assume that all vertices in $B_i \setminus v_1$ are in $V_1(v_1)$. In particular, $v_2 \in V_1(v_1)$ and hence $v_1 \in \overline{V_1}(v_2)$. Similarly, since v_2 is a cut vertex, all vertices of B_i lie in the same (designated) cone with apex v_2. Since $v_1 \in \overline{V_1}(v_2)$, all vertices in $B_i \setminus v_2$ are in $\overline{V_1}(v_2)$. If v_3 is a vertex in B_i, distinct from v_1 and v_2, then from the discussion above, we get $v_3 \in V_1(v_1)$ and $v_3 \in \overline{V_1}(v_2)$. Hence $v_1 \in \overline{V_1}(v_3)$ and $v_2 \in V_1(v_3)$. Suppose v_3 is a cut vertex in $G_{\davidsstar}(P)$. Since v_1 and v_2 are in the same connected component of $G_{\davidsstar}(P) \setminus v_3$, it is a contradiction to Lemma 6, that $v_1 \in \overline{V_1}(v_3)$ and $v_2 \in V_1(v_3)$.

Thus, if the node corresponding to B_i is not a leaf node of the block cut point graph of $G_{\davidsstar}(P)$, then exactly two vertices in B_i are cut vertices of $G_{\davidsstar}(P)$. Since no three blocks can share a common vertex by Lemma 6, we are done. □

6.2. *Number of Edges of $G_{\davidsstar}(P)$*

Since $G_{\bigtriangledown}(P)$ and $G_{\bigtriangleup}(P)$ are planar graphs and $G_{\davidsstar}(P) = G_{\bigtriangledown}(P) \cup G_{\bigtriangleup}(P)$, using Euler's theorem, it is obvious that $G_{\davidsstar}(P)$ has at most $2 \times (3n-6) = 6n-12$ edges, where $n = |P|$ [12]. In this section, we show that for any point set P, its $G_{\davidsstar}(P)$ has a spanning tree of a special structure, which will imply that $G_{\davidsstar}(P)$ can have at most $5n-11$ edges.

Lemma 7. *For a point set P, the intersection of $G_{\bigtriangledown}(P)$ and $G_{\bigtriangleup}(P)$ is a connected graph.*

PROOF. We will prove this algorithmically. At any point of execution of this algorithm, we maintain a partition of P into two sets S and $P \setminus S$ such that the induced subgraph of $G_{\bigtriangledown}(P) \cap G_{\bigtriangleup}(P)$ on S is connected. When the algorithm terminates, we will have $S = P$, which will prove the lemma.

We start by adding any arbitrary point $p_1 \in P$ to S. The induced subgraph of $G_{\bigtriangledown}(P) \cap G_{\bigtriangleup}(P)$ on S is trivially connected now.

At any intermediate step of the algorithm, let $S = \{p_1, p_2, \ldots, p_k\} \neq P$, such that the invariant is true. We will show that we can add a point p_{k+1} from $P \setminus S$ into S, and still maintain the invariant.

For any point $p \in S$, let

$$d_1(p) = \min_{i \in \{1,2,3\}, p' \in V_i(p) \cap P \setminus S} c_i(p, p')$$

$$d_2(p) = \min_{i \in \{1,2,3\}, p' \in \overline{V_i}(p) \cap P \setminus S} \overline{c_i}(p, p')$$

and

$$d(p) = \min(d_1(p), d_2(p))$$

Since $|P \setminus S| \geq 1$, $d(p) < \infty$. Let $d = \min_{p \in S} d(p)$.

Consider $p \in S$ such that $d(p) = d$. By definition of d, such a point exists. Consider the area enclosed by the hexagon around p which is defined by $H_p = \bigcup_{i=1}^{3} \{p' \in C_i(p) \mid c_i(p, p') \leq d\} \cup \bigcup_{i=1}^{3} \{p' \in \overline{C_i}(p) \mid \overline{c_i}(p, p') \leq d\}$. (See

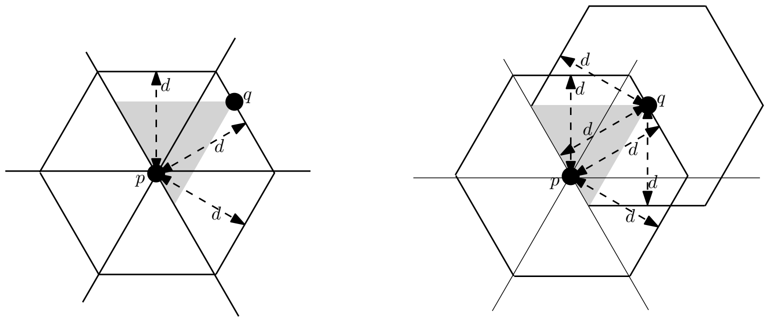


Figure 12: (a) Closest point to p. (b) Hexagons around closest pairs.

Figure 12 (a)). We know that there exists a point $q \in P \setminus S$ such that q is on the boundary of H_p. We claim that pq is an edge in $G_\triangledown(P) \cap G_\triangle(P)$.

Let $H_q = \bigcup_{i=1}^{3} \{p' \in C_i(q) \mid c_i(q,p') \le d\} \cup \bigcup_{i=1}^{3} \{p' \in \overline{C_i}(q) \mid \overline{c_i}(q,p') \le d\}$, which is a hexagonal area around q. (See Figure 12 (b)). Without loss of generality, assume that $q \in C_1(p)$. Note that, by Property 1, $c_1(p,q) = \overline{c_1}(q,p) = d$ and hence, $\triangledown pq \cup \triangle pq \subseteq H_p \cap H_q$.

If there exists a point $q' \in (P \setminus \{q\}) \setminus S$ such that q' lies in the interior of H_p, then $d(p) < d$, which is a contradiction. Similarly, if there exists a point $p' \in (P \setminus \{p\}) \cap S$ such that p' lies in the interior of H_q, then $d(p) < d$. This is also a contradiction. Therefore, $H_p \cap H_q \cap (P \setminus \{p,q\}) = \emptyset$. Since, $\triangledown pq \cup \triangle pq \subseteq H_p \cap H_q$, this implies that $\triangledown pq \cap (P \setminus \{p,q\}) = \emptyset$ and $\triangle pq \cap (P \setminus \{p,q\}) = \emptyset$. This implies that pq is an edge in $G_\triangledown(P)$ as well as in $G_\triangle(P)$.

Since pq is an edge in $G_\triangledown(P) \cap G_\triangle(P)$, we can add $p_{k+1} = q$ to the set S, thus increasing the cardinality of S by one, and still maintaining the invariant that the induced subgraph of $G_\triangledown(P) \cap G_\triangle(P)$ on S is connected. Since we can keep on doing this until $S = P$, we conclude that $G_\triangledown(P) \cap G_\triangle(P)$ is connected. □

Theorem 5. *For a set P of n points in general position, $G_{\text{✡}}(P)$ has at most $5n-11$ edges and hence its average degree is less than 10.*

PROOF. Since $G_\triangledown(P)$ and $G_\triangle(P)$ are both planar graphs we know that each of them can have at most $3n-6$ edges. From Lemma 7, we know that the intersection of $G_\triangledown(P)$ and $G_\triangle(P)$ contains a spanning tree and hence they have at least $n-1$ edges in common. From this, we conclude that the number of edges in $G_{\text{✡}}(P) = G_\triangledown(P) \cup G_\triangle(P)$ is at most $(3n-6)+(3n-6)-(n-1) = 5n-11$. Hence,the average degree of $G_{\text{✡}}(P)$ is less than 10. □

Corollary 2. *For a set P of n points in general position, its Θ_6 graph has at most $5n-11$ edges.*

It is still an open problem to decide whether the upper bound on the number of edges, stated in Theorem 5 and Corollary 2, is tight. Here we give an example showing that this upper bound cannot be improved below $\left(4+\frac{1}{3}\right)n-13$. In Figure 13, a point set P of 18 points and the corresponding $G_{\text{✡}}(P)$ graph is shown. This graph has 65 edges. By varying the number of triplets of points (a_i,b_i,c_i), $i \ge 0$, in P, following the same pattern, we can show that for any $n \ge 6$ which is a multiple of 3, there is a point set P of n points in general position, such that $G_{\text{✡}}(P)$ has exactly $\left(4+\frac{1}{3}\right)n-13$ edges.

7. Conclusions

We have shown that for any set P of n points in general position, any maximum $\triangledown$ (resp. $\triangle$) matching of P will match at least $2\left(\left\lceil\frac{|P|-1}{3}\right\rceil\right)$ points. This also implies that any half-Θ_6 graph for point sets in general position has a matching of size at least $\left\lceil\frac{|P|-1}{3}\right\rceil$. We have also given examples for which this bound is tight. We also proved that when P is in general position, the block cut point graph of its Θ_6 graph is a simple path and that the Θ_6 graph has at

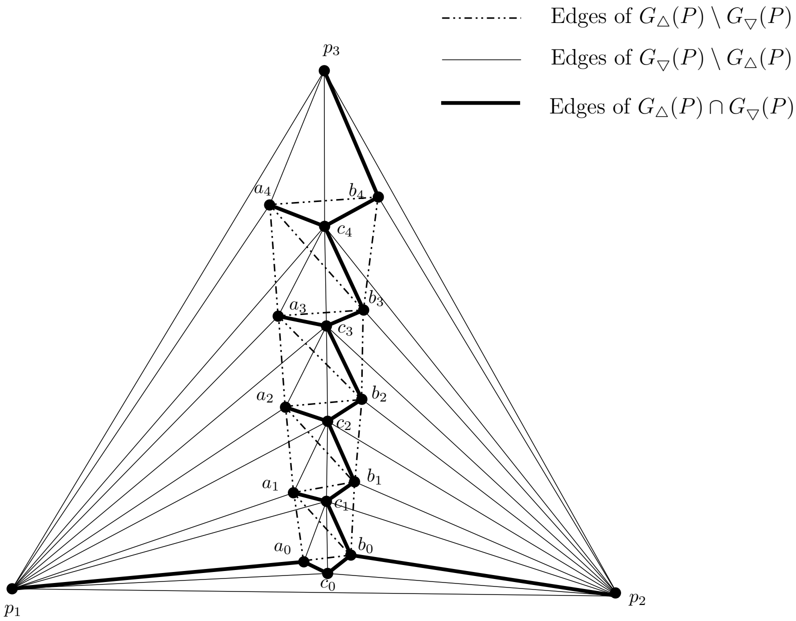


Figure 13: A point set P of $n = 18$ points and the corresponding $G_{✡}(P)$ graph with $\left(4+\frac{1}{3}\right)n-13 = 65$ edges.

most $5n-11$ edges. It is an interesting question to see whether for every point set in general position, its Θ_6 graph contains a matching of size $\left\lfloor \frac{|P|}{2} \right\rfloor$. So far, we were not able to get any counter examples for this claim and hence we conjecture the following.

Conjecture 1. *For every set of n points in general position, its* Θ_6 *graph contains a matching of size* $\left\lfloor \frac{n}{2} \right\rfloor$.

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