APPROXIMATION TO TRUTH AND THEORY OF ERRORS[1]

Rolando Chuaqui[2] Leopoldo Bertossi[3]
Universidad Católica de Chile
Casilla 114-D
Santiago, Chile

1. Introduction.

The purpose of this paper is twofold. On the one hand, we shall give in Section 2, a model for the theory of errors based on the probability structures of Chuaqui 1983 and 1984. The construction of this model uses recent result in non-standard analysis. On the other hand, a formalization of the notion of "approximation to truth", which includes the possibility of random errors in measurement, will be presented in Section 3. This last section is mainly the work of the first author, who takes full responsability for it.

A formalization of the notion of approximation to truth was presented in a paper inspired by ideas of N.da Costa(Mikenberg, da Costa, and Chuaqui 198+, MDC, for short), but there, the possibility of random errors in measurement was not taken into account. As an introduction to our ideas in the present paper, we shall briefly summarize the formalization of MDC.

A relational structure $\mathfrak{A} = \langle A, R_i \rangle_{i \in I}$ is thought of as a theoretical physical structure about the objects in A. We think of this total structure (i.e. for each n-tuples of elements of A, it is determined whether it belongs to the n-ary relation R_i or not) as what the theory gives us for the objects in A. A natural way to formalize this point of view, is to consider scientific theories as set-theoretical predicates, as in Suppes 1957, Chapter 12. What is actually known about the objects in A, is in the <u>partial structure</u> $\mathfrak{A}' = \langle A, R_i' \rangle_{i \in I}$, where each R_i' is only a partially defined relation over elements of A. That is, only for some tuples it is determined whether they belong R_i or not. For the rest of the tuples, it is undetermined. A theoretical structure $\mathfrak{A}$ is adequate for (or compatible with) $\mathfrak{A}'$, if $\mathfrak{A}$ is an extension of $\mathfrak{A}'$, i.e. R_i coincides with R_i', where the latter is defined. We define a sentence to be true in $\mathfrak{A}'$, if it is true in all extensions with the same universe A. A theoretical structure adequate for $\mathfrak{A}'$ is one of these extensions, $\mathfrak{A}$, i.e. what is true in $\mathfrak{A}'$ is also true in $\mathfrak{A}$, and nothing that is false in $\mathfrak{A}'$ is true in $\mathfrak{A}$. Thus, there may be several theoretical structures which are adequate for $\mathfrak{A}'$, each one determined (possibly) by different theories. When our knowledge about the objects in A increases we obtain another partial structure $\mathfrak{A}''$, which is an extension of $\mathfrak{A}'$. If $\mathfrak{A}$ were compatible with $\mathfrak{A}'$, but not with $\mathfrak{A}''$, then it must be changed and the corresponding theory also.

Another possibility for the rejection of the theory is the following. Suppose that $\mathfrak{A}'$ is what we know for the objects in A and that a theory T, say classical mechanics, determines a total structure $\mathfrak{A}$, with universe (or domain) A, that is compatible with $\mathfrak{A}'$. Assume that if we take a different domain B (which may include A), then $\mathfrak{B}'$ is what we know about the objects in B, and that the theory T determines, for B, a total structure $\mathfrak{B}$ that is incompatible with $\mathfrak{B}'$. Then T should be rejected, but we would say that it is still approximately true for the objects in A. For instance, if we take A as the medium sized objects at slow velocities and T as classical mechanics, then T is approximately true for A.

This picture, however, is not completely accurate. The theoretical structure $\mathfrak{A}$, given by T, in general does not extend the partial structure $\mathfrak{A}'$ exactly. We usually say that $\mathfrak{A}$ coincides with $\mathfrak{A}'$, except for possible errors in measurement. It is this last factor that we want to formalize in the present paper.

In the measurement of a certain quantity, we usually assume that there is a theoretical value, given in the total structure $\mathfrak{A}$, but that the actual value, obtained,i.e. occurring in $\mathfrak{A}'$, may differ from it because of errors in the procedure of measurement. There are three main sources for this error. Sometimes, there is a systematic error derived from the procedure itself. Second, there may be an error produced by the limit of precision of the measurement method. Lastly there may be random errors. We shall disregard the first two types of error and consider just random errors, which, in general are the most important. The account presented in Section 2 could be modified so as to take into consideration the other types of errors.

Thus, if we include random errors in measurement, then the theoretical structure $\mathfrak{A}$ might not be an extension of the partial structure $\mathfrak{A}'$, but, anyway, be considered compatible with it. That is, there might be a sentence true in $\mathfrak{A}'$, but false, strictly speaking, in $\mathfrak{A}$, without this fact being enough ground to reject $\mathfrak{A}$ (and hence T). In order to formalize this situation, we do the following. In the first place, we associate with $\mathfrak{A}$ another structure that we call an error-structure $\mathfrak{A}_e$. Now, a sentence ϕ, instead of being true or false in $\mathfrak{A}_e$, has a probability $P_{\mathfrak{A}_e}(\phi)$, which depends on $\mathfrak{A}_e$. If there is no sentence true in $\mathfrak{A}'$ that has low probability according to $\mathfrak{A}_e$, and high probability according to $\mathfrak{B}_e$, where $\mathfrak{B}$ is an alternative theoretical structure of $\mathfrak{A}$, then $\mathfrak{A}$ is compatible with $\mathfrak{A}'$.

Probability is thought of as "degree of partial truth" (see Chuaqui 1977, for a justification for this view). Because of the possibility of error, we cannot get to truth; thus, we should strive to get as close as possible to it, i.e. to high probability. In a similar way, a false consequence means rejection, but we cannot get to falsehood, but just to low probability, which is approximate falsehood. In fact, we should try to approximate falsehood as much as possible, i.e. given any $\varepsilon > 0$, to try to get a sentence ϕ true in $\mathfrak{A}'$ but with $P_{\mathfrak{A}_e}(\phi) < \varepsilon$. In Chuaqui 198+, there is a discussion of how to approximate falsehood with a sequence of probabilities decreasing to zero. In order to do this, we need sequences of trials of the same experiment. Thus, our structures $\mathfrak{A}_e$, have to be complicated somewhat for this purpose: we construct from $\mathfrak{A}_e$, the structure $\mathfrak{A}_e^{\omega}$ that formalizes an unlimited number of repetitions of the experiments and define probabilities for sentences according to $\mathfrak{A}_e^{\omega}$, derived from $P_{\mathfrak{A}_e}$. These probabilities are defined using the methods of Chuaqui 198+'

This presentation is offered, not as a program for practical implementation, but only as a way to illuminate the relations between theory and evidence, and between truth and probability, in science. It is clear, that the models presented here are a preliminary version that is oversimplified. In particular, we just consider deterministic theories. We hope to improve these models in the future, and include nondeterministic theories.

2. A theoretical model for the theory of errors.

In this section, we present a model for the theory of random errors. In this theory, a quantity is supposed to have a theoretical value, but the procedure of measurement introduces a random error that comes from a combination of a large number of independent causes, each one producing a very small error in the positive or negative direction. The total error for a particular measurement is obtained by adding up the errors produced by the different causes.

In the compound probability structures of Chuaqui 1983, the causes are represented by a causal tree T, with a partial ordering relation. Since here the causes are independent, two different elements of T are never related, i.e. one has no influence upon the other. The fact that there is a large number of causes, can be represented, in non standard analytic terms, by taking T to be the internal set $T = \{t_o, t_1, \ldots, t_\eta\}$ where η is a non-standard infinite natural number, i.e. $\eta \in {}^*\mathbb{N} \sim \mathbb{N}$. That is, T is an infinite, but hyperfinite internal set. With each cause t_k we associate a positive infinitesimal number ε_k and t_k may cause the error ε_k or $-\varepsilon_k$ with equal probability. In order to describe the action of t_k, we introduce a simple probability structure K_k, as in Chuaqui 1984. K_k consists of two relational structures, namely,

$$K_k = \{ \langle \{-\varepsilon_k, \varepsilon_k\}, \{-\varepsilon_k\} \rangle , \langle \{-\varepsilon_k, \varepsilon_k\}, \{\varepsilon_k\} \rangle \}$$

The universe of K_k is $\{-\varepsilon_k, \varepsilon_k\}$. The first structure obtains, if at t_k, $-\varepsilon_k$ is produced, and the second, if ε_k is. The algebra of events naturally consists of all subsets of K_k. In order to obtain the probability measure μ_k, we need a group of permutations of $\{-\varepsilon_k, \varepsilon_k\}$, G_{K_k}. In this case, it clearly contains all permutations of this set and μ_k, the G_{K_k} - invariant measure, assigns 1/2 to each of the models of K_k.

The probability structure for the action of all causes is a compound structure with causal tree $\langle T, = \rangle$. In order to keep the total error with bounds, we must assume that $\sum_{k=0}^{\eta} \varepsilon_k^2 = \varepsilon^2$ where ε is a finite positive number.

The set of compounds outcomes is

$$H = \Pi \langle K_k : t_k \in T \rangle ,$$

i.e., H consists of the functions ξ with domain T and such that $\xi(t_k) \in K_k$ for each $k \leq \eta$. Each $\xi \in H$, represents a possible measurement (we assume that the theoretical value to be measured is 0). As in Chuaqui 1983 the probability measure μ defined on subsets of H is the product measure of the μ_k for $k < \eta$.

The result of the measurement represented by an outcome $\xi \in H$ is given as follows. For each $k \leq \eta$, we first define a random variable $X_k \to {}^*\mathbb{R}$ (the non-standard reals or hyperreal numbers) by,

$$X_k(\xi) = \delta_k, \quad \text{if} \quad \xi(t_k) = \langle \{\varepsilon_k, -\varepsilon_k\} , \{\delta_k\} \rangle$$

(i.e., $\delta_k = \varepsilon_k$ or $\delta_k = -\varepsilon_k$).

Then, the result of the measurement in outcome $\xi \in H$ is

$$f(\xi) = \sum_{k=0}^{\eta} {}_k(\xi)$$

In order to study the distribution of f, we need the central limit theorem with the Lindeberg condition in non-standard form as given in Stoll 1982:

If $\eta \in {}^*\mathbb{N} - \mathbb{N}$ and $< Y_k : k \leq \eta >$ is an internal sequence of *independent random variables in an internal probability space (Ω, A, μ) such that $E(Y_k) = 0$ and $E(Y_k{}^2) = 1$, and $< \alpha_k : k \leq \eta >$ is a sequence of infinitesimal weights $\alpha_k \in {}^*\mathbb{R}$ such that $\sum_{k=0}^{\eta} \alpha_k{}^2 = \alpha^2$ with $0 < {}^\circ\alpha \in \mathbb{R}^+$, then

$$\mu([\sum_{k=0}^{\eta} \alpha_k Y_k \leq \lambda]) \simeq {}^*\Phi(\tfrac{\lambda}{\alpha}) \text{ for all } \lambda \in {}^*\mathbb{R}.$$

Here $^*\Phi$ is the non-standard normal distribution with mean 0 and standard deviation 1.

Taking standard parts, one can show that if γ is Loeb's measure generated by ${}^\circ\mu$, then

$$\gamma([{}^\circ \sum_{k=0}^{\eta} \alpha_k Y_k \leq \lambda]) = \Phi(\tfrac{\lambda}{{}^\circ\alpha}),$$ for all $\lambda \in \mathbb{R}$, where ϕ is the standard normal distribution.

This work of Stoll is based on Loeb 1975 and Anderson 1976.

Thus, we have for our random variable $f : H \to {}^*\mathbb{R}$, the following distribution:

$$\mu([f \leq \lambda]) = \mu([\sum_{k=0}^{\eta} \varepsilon_k \left[\frac{X_k}{\varepsilon_k}\right] \leq \lambda])$$

$$\simeq {}^*\phi(\tfrac{\lambda}{{}^\circ\varepsilon}),$$

for each $\lambda \in {}^*\mathbb{R}$.

Then, its standard distribution is,

$$\gamma([{}^\circ f \leq \lambda]) = \phi(\tfrac{\lambda}{{}^\circ\varepsilon}),$$

for each $\lambda \in \mathbb{R}$

Thus, the result of the measurement is normally distributed with mean 0 and standard deviation ${}^\circ\varepsilon$. It is easy to modify the construction so that the mean (i.e. the theoretical measurement) is any number $r \in \mathbb{R}$. Thus, in order to obtain a model for the measurement of a certain quantity, we must be given two parameters: the theoretical measurement r and the standard error σ_ε. With these two numbers, we construct a compound probability structure as above (that will be called an error probability structures). Then the random variable f, which gives the actual value of the measurement, will be normally distributed with mean r and standard deviation σ_ε.

The mean r represents the theoretical measurement and may be obtained by calculations from the theory or be estimated from the data. For instance, a measurement of length is usually obtained directly from the data, but other quantities may be consequences or the theory.

The standard deviation σ_ε, on the other hand, depends on the method of measurement. If the method is more accurate ε° will be smaller, i.e. there may be less causes of error of the error produced by each cause may be smaller. There may be several procedures of measurement for the same quantity all should give the same mean, but possibly have different standard deviations. This standard error is usually estimated from the distribution of actual measurements, but occasionally it is roughly estimated from theoretical considerations concerning the supposed precision of the procedure.

These values of the mean and standard error are, then, compared with a series of actual values using the usual statistical techniques. If the distribution of the actual values is very improbable according to the theoretical distribution then this last one is rejected as a model of the real state of affairs.

Although the mean has a theoretical significance, the standard error has not, since we are just considering deterministic phenomena. Thus, it is usually important just to test the appropriateness of the theoretical value of the measurement as compared with the actual values obtained. Which is the theoretical standard error (i.e. the standard deviation of the error probability structure) is not important. A statistical test for testing the mean with unknown standard deviation is Student's Test. For using this test, we calculate the quantity

$$t = \frac{M - r}{S_M} ,$$

when r is the theoretical value of the measurement, M the mean value of the values actually obtained, and S_M the standard deviation of these values. We can then compute the probability of $|t| < a$ for an $a \in \mathbb{R}$, for any error probability structure with mean r. We can obtain a value of a for which these probabilities (one for each error probability structure) are less than a certain α. Thus, whatever may the standard deviation of the error probability structure be, the probability of

$|t|<a$, is less than α. If we take α sufficiently small, then the probability of the event $|t|<a$, will be small for r being the theoretical value of the measurement. Here, the hypothesis that r is this value, is to be rejected.

3. Structures with errors in measurement.

We are now ready to introduce a theoretical structure that includes measurement with random errors, and the corresponding partial structure representing what we actually know. In fact, we shall introduce two new types of theoretical structures : a pure measurement structure or $\mathbb{R}$-structure, and the error structures associated with it. We shall discuss later the structures that represent our knowledge and which correspond to the partial structures of the old setting sketched in Section 1.

An $\mathbb{R}$-structure is a system of the form

$$\mathfrak{A} = \langle A, f_i^{\mathfrak{A}}, R_j^{\mathfrak{A}} \rangle_{i\in I, j\in J},$$

where each $R_j^{\mathfrak{A}}$, for $j\in J$, is an n_j-ary relation between elements of A, and each $f_i^{\mathfrak{A}}$, for $i\in I$, is an n_i-ary operation from A into $\mathbb{R}$. We could also have operations from A into A, or distinguished elements of A, but, for simplicity, we shall not include them, since they can be replaced by relations. We think of $\mathfrak{A}$ as what we accept theoretically to be true of the elements of A. The $R_j^{\mathfrak{A}}$'s represent possible relations between these elements, and the $f_i^{\mathfrak{A}}$'s, measurements performed on them. Thus, $f_i^{\mathfrak{A}}(a_0,\ldots,a_{n_i-1})$ is a real number that measures some property of the system $(a_0,\ldots,a_{n-1})$ of elements of A. There could be two measurements $f_i^{\mathfrak{A}}$ and $f_k^{\mathfrak{A}}$, with $i\neq k$, of the same quantity. In the $\mathbb{R}$-structure $\mathfrak{A}$ they could coincide. However, in the error structures and the partial structures to be introduced below, they may differ. As in Section 1, $\mathfrak{A}$ is what the scientific theory prescribes for the elements of A. In this paper, we only consider deterministic theories.

The language for $\mathbb{R}$-structures is a one-sorted language, with variables $x,y,z,\ldots$, that constains the following types of atomic formulas:

$x = y$

$R_j\, x_0,\ldots,x_{n_j-1}$ for each $j\in J$,

and

$[f_i(x_0,\ldots,x_{n_i-1}) \geq r]$, for each $r\in\mathbb{Q}$ (the rational numbers), and each $i\in I$.

These formulas are combined in an $L_{\omega_1\omega}$-language with negation, countable conjunctions and disjunctions, and finitely many quantifiers.

The variables are assigned elements of A. For each formula ϕ and each assigment s of the variables in A, we define when s satisfies ϕ in $\mathfrak{A}$, in symbols $\mathfrak{A} \models \phi[s]$. Most of the clauses are the usual ones, plus

$$\models [f_i(x_o,\ldots,x_{n_i-1}) \geq r][s] \quad \text{iff} \quad f_i^{\mathfrak{A}}(s(x_o),\ldots,x(x_{n_i-1})) \geq r.$$

We could also have a two-sorted language with variables and operations for the real numbers, but we shall not need this in this paper.

Most of the mathematical results obtained in MDC could easily be extended to $\mathbb{R}$-structures, with the natural notion of partial $\mathbb{R}$-structure. We shall not pursue this line here. Instead, we shall introduce another type of theoretical structure: the error-structures (or briefly, E-structures) associated with the $\mathbb{R}$-structure $\mathfrak{A}$. While $\mathfrak{A}$ determines whether a formula is satisfied by an assignment or not, an E-structure $\mathfrak{A}_e$, determines only the probability that is assigned to the formula.

An <u>E-structure associated with</u> $\mathfrak{A}$ is a system of the form

$$\mathfrak{A}_e = \langle A, f_i^{\mathfrak{A}_e}, R_j^{\mathfrak{A}_e} \rangle_{i \in I, j \in J},$$

where $R_j^{\mathfrak{A}_e} = R_j^{\mathfrak{A}}$, for $j \in J$, and, if $f_i^{\mathfrak{A}}$ is an n_i-ary operation, then for each $a_o,\ldots,a_{n_i} \in A$, $f_i^{\mathfrak{A}_e}(a_o,\ldots,a_{n_i-1})$ is a random variables whose distribution is given by an error probability structure (see Section 2); $f_i^{\ e}(a_o,\ldots,a_{n_i-1})$ is a random variable having a normal distribution with mean $f_i^{\mathfrak{A}}(a_o,\ldots,a_{n_i-1})$, for each $i \in I$. More precisely, $f_i^{\mathfrak{A}_e}(a_o,\ldots,a_{n_i-1}) = f_i^{\mathfrak{A}}(a_o,\ldots,a_{n_i-1}) + \varepsilon_i(a_o,\ldots,a_{n_i-1})$, where $\varepsilon_i(a_o,\ldots,a_{n_i-1})$ is random variable with mean 0, which represents the error in the measurement. The distribution of $\varepsilon_i(a_o,\ldots,a_{n_i-1})$ is determined by an error probability structure (of Section 2) whose universe H is the domain of $\varepsilon_i(a_o,\ldots,a_{n_i-1})$.

Each $f_i^{\mathfrak{A}}$, for $i \in I$, represents one method of measurement for a quantity. There may be several methods for the same quantity indexed by different elements of I. Thus, $f_i^{\mathfrak{A}}$ and $f_k^{\mathfrak{A}}$ may be measurements of length, say by a ruler and by wavelengths. In this case, $f_i^{\mathfrak{A}}(a_o,\ldots,a_{n-1}) = f_k^{\mathfrak{A}}(a_o,\ldots,a_{n-1})$ for every $a_o,\ldots,a_{n-1} \in A$. Hence, $f_i^{\mathfrak{A}_e}(a_o,\ldots,a_{n-1})$ and $f_k^{\mathfrak{A}_e}(a_o,\ldots,a_{n-1})$ are random variables with the same mean. But, their standard deviations may be different. This deviation depends on the procedure of measurement, and may be determined by the theory of the method or the empirical data. as was explained in Section 2.

It may be more reasonable to assume that for each $i \in I$, $f_i^{\mathfrak{A}_e}$ is defined only for a subset of A, namely, for those objects that are possible to measure with the procedure involved. This would introduce additional inessential complications to our models, so that we shall assume that for each tuple $a_o, \ldots, a_{n-1}$ of elements of A, $f_i^{\mathfrak{A}_e}(a_o, \ldots, a_{n-1})$ is defined (i.e. it is a random variable with values in $\mathbb{R}$).

The language for the E-structures $\mathfrak{A}_e$ associated with $\mathfrak{A}$, is the same as that for $\mathfrak{A}$. However, instead of satisfaction, $\mathfrak{A}_e$ determines a probability. For each formula ϕ and each assignment s of the variables in A, we define the probability that s assigns to ϕ in $\mathfrak{A}_e$, in symbols $P_{\mathfrak{A}_e}(\phi, s)$. The definition that will be given below is based in that of Scott and Krauss 1966. The main differences with Scott and Krauss are that we use assignments instead of constants, and that we give the definition jointly for all relations and operations instead of doing it separately, and then joining them by their method of independent unions. In any case, this means that we assume the different $f_i^{\mathfrak{A}_e}$ and R_j, for $i \in I$ and $j \in J$, to be stochastically independent.

We now proceed to state the definition of $P_{\mathfrak{A}_e}$ in several stages. Let $\mathcal{B}_i^{\mathfrak{A}_e}(a_o, \ldots, a_{n-1})$ be the measure algebra of the error probability structure where $f_i^{\mathfrak{A}_e}(a_o, \ldots, a_{n-1})$ is defined, and $\mu_i^{\mathfrak{A}_e}(a_o, \ldots, a_{n-1})$, its measure. We shall always consider, now and in what follows, strictly positive measures, i.e. measures that vanish only on the zero of the algebra, and their corresponding measure algebras. This is needed because in our definitions we must have complete algebras, i.e. algebras where the suprema and infima are always defined. If necessary, to achieve a strictly positive measure, we take the algebra (and the measure) modulo its null sets. In what follows, we shall suppose that this is done, without mentioning it.

For each $i \in I$, $\mathcal{B}_i^{\mathfrak{A}_e}$ is defined to be the product algebra.

$$\mathcal{B}_i^{\mathfrak{A}_e} = \Pi < \mathcal{B}_i^{\mathfrak{A}_e}(a_o, \ldots, a_{n-1}) : a_o, \ldots, a_{n-1} \in A > .$$

and $\mu_i^{\mathfrak{A}_e}$ its corresponding (strictly positive) measure.

For each $j \in J$, $\mathcal{B}_j^{\mathfrak{A}_e}$ is the two element measure algebra $\{0_j, 1_j\}$, and $\mu_j^{\mathfrak{A}_e}$ the measure that assigns 1 to 1_j and 0 to 0_j.

We consider, now, the product $\mathcal{B}^{\mathfrak{A}_e}$ of all these algebras:

$$\mathcal{B}^{\mathfrak{A}_e} = \Pi < \mathcal{B}_k^{\mathfrak{A}_e} : k \in I \cup J >,$$

and its corresponding product measure, $\mu^{\mathfrak{A}_e}$.

An element in $\mathcal{B}^{\mathfrak{A}_e}$ is a system

$$\mu = < \mu_k : k \in I \cup J > ;$$

on its turn, if $i \in I$, then μ_i is a system

$$\mu_i = < \mu_i (a_o, \ldots, a_{n_i - 1}) : a_o, \ldots, a_{n_i - 1} \in A >$$

We call the unit of $\mathcal{B}^{\mathfrak{A}_e}$, $\mathbb{1}$, and its zero $\mathbb{0}$. Similarly, $1_k, 0_k$ will be the corresponding elements of $\mathcal{B}_k^{\mathfrak{A}_e}$, for $k \in I \cup J$, where, if $i \in I$, $1_i = < 1_i(a_o, \ldots, a_{n_i - 1}): a_o, \ldots, a_{n_i - 1} \in A >$, and $0_i = < 0_i(a_o, \ldots, a_{n_i - 1}) : a_o, \ldots, a_{n_i - 1} \in A >$.

For each formula ϕ and assignment s of the variables in A, we define a valuation $h(\phi; s) \in \mathcal{B}^{\mathfrak{A}_e}$, by recursion:

(i) $$h(x=y;s) = \begin{cases} \mathbb{1}, & \text{if } s(x) = s(y) \\ \mathbb{0}, & \text{otherwise} \end{cases}$$

(ii) $h(R_j x_o, \ldots, x_{n-1}, s) = \mu$, where its components μ_j are given by

$$\mu_j = \begin{cases} 1_j, & \text{if } < s(x_o), \ldots, s(x_{n-1} 0 > \in R_j^{\mathfrak{A}} \\ 0_j, & \text{otherwise,} \end{cases}$$

and $\mu_k = 1_k$, for all $k \in I \cup J$ with $j \neq k$.

(iii) $h([f_i(x_o, \ldots, x_{n-1}) \geq r] ; s) = \mu$, where

$$\mu_i(s(x_o), \ldots, s(x_{n-1})) = [f_i^{\mathfrak{A}_e}(s(x_o), \ldots, s(x_{n-1})) \geq r]$$

(i.e. the corresponding element of $\mathcal{B}_i^{\mathfrak{A}_e}(s(x_o), \ldots, s(x_{n-1}))$,

$$\mu_i(a_o, \ldots, a_{n-1}) = 1_i(a_o, \ldots, a_{n-1}) \text{ for } (a_o, \ldots, a_{n-1}) \neq$$

$(s(x_o), \ldots, s(x_{n-1}))$, and

$$\mu_k = 1_k, \text{ for } k \in I \cup J, \ k \neq i.$$

(iv) $h(\neg \phi; s) = \mathbb{1} - h(\phi, s)$

(v) $h(\bigvee_{n \in \mathbb{N}} \phi_n; s) = \bigvee_{n \in \mathbb{N}} h(\phi_n, s)$

(vi) $h(\bigwedge_{n \in \mathbb{N}} \phi_n; s) = \bigwedge_{n \in \mathbb{N}} h(\phi_n, s)$

(vii) $h(\exists x\phi;s) = \bigvee_{a\in A} h(\phi,s^x_a)$

(viii) $h(\forall_x x\phi;s) = \bigwedge_{a\in A} h(\phi,s^x_a)$

Here, s^x_a is the assigment that coincides with s everywhere, except, possibly, on x where it assigns a.

Now we are ready to define $P_{\mathfrak{A}_e}(\phi,s)$, the probability that s assigns $\mathfrak{A}_e$. This is simply given by

$$P_{\mathfrak{A}_e}(\phi,s) = \mu^{\mathfrak{A}_e}(h(\phi;s)).$$

We shall now proceed to the discussion of the structures that represent what we actually know and their relation to the theoretical structures. In order to study this relationship, it is not enough to consider one $\mathbb{R}$-structure $\mathfrak{A}$, but need to consider all of its alternatives, as well. An <u>alternative to</u> the $\mathbb{R}$-structure $\mathfrak{A}$ is an $\mathbb{R}$-structure $\mathfrak{B}$, with the same universe A and the same similarity type. That is, if $\mathfrak{A} = \langle A, f_i^{\mathfrak{A}}, R_j^{\mathfrak{A}} \rangle_{i\in I, j\in J}$, then $\mathfrak{B} = \langle A, f_i^{\mathfrak{B}}, R_j^{\mathfrak{B}} \rangle_{i\in I, j\in J}$ where $f_i^{\mathfrak{B}}$ and $R_j^{\mathfrak{B}}$ are of the same arity as $f_i^{\mathfrak{A}}$ and $R_j^{\mathfrak{A}}$, respectively (The similarity type τ determines for each $k\in I\cup J$, where the symbol indexed by k is an operation or a relation, and its arity). The set of alternatives with universe A and similarity type τ, we call the A,τ-alternatives.

For each alternative $\mathfrak{B}$ to $\mathfrak{A}$, we construct the corresponding E-structure. If $\mathfrak{A}_e$ is an E-structure associated with $\mathfrak{A}$, we shall designate by $\mathfrak{B}_e$, the E-structure associated to $\mathfrak{B}$ in which the distribution of $f_i^{\mathfrak{B}_e}(a_0,\ldots,a_{n-1})$ has the same standard deviation as that of $f_i^{\mathfrak{A}_e}(a_0,\ldots,a_{n-1})$.

Now, we define an A,τ-partial $\mathbb{R}$-structure $\mathfrak{A}'$, where A is a universe (i.e. a nonempty set) and τ a similarity type. $\mathfrak{A}'$ is a system of the form:

$$\mathfrak{A}' = \langle A, f_i^{\mathfrak{A}'}, R_j^{\mathfrak{A}'} \rangle_{i\in I,\ j\in J}$$

where $f_i^{\mathfrak{A}'}$, for $i\in I$, is an n_i-ary partial function from A into $\mathbb{R}$, and $R_j^{\mathfrak{A}'}$, for $j\in J$, is an n_j partial relation. For describing partial structures, it is better to replace relations by their characteristic functions, i.e. we write

$$R_j^{\mathfrak{A}'}(a_0,\ldots,a_{n_j-1}) = 1, \text{ if } \langle a_0,\ldots,a_{n_j-1}\rangle \in R_j^{\mathfrak{A}'}$$
$$= 0, \text{ otherwise}$$

Then, a partial relation is a partial function from A into $\{0,1\}$.

A complete extension $\mathfrak{B}$ of the A,τ-partial $\mathbb{R}$-structure $\mathfrak{A}'$ is an A,τ-alternative (i.e. an $\mathbb{R}$-structure with universe A and similarity type τ) such that the operations and relation of $\mathfrak{B}$ are extension of those in $\mathfrak{A}'$. We already have defined satisfaction for $\mathbb{R}$-structures. We can now define satisfaction for partial $\mathbb{R}$-structures $\mathfrak{A}'$ as in MDC, namely, for any formula ϕ, and assignment s in A:

$\mathfrak{A}' \models_T \phi[s]$ iff for every complete extension $\mathfrak{B}$ of $\mathfrak{A}'$, we have

$$\mathfrak{B} \models \phi[s]$$

$\mathfrak{A}' \models_F \phi[s]$ iff $\mathfrak{A}' \models_T \neg\phi[s]$

$\mathfrak{A}' \models_U \phi[s]$, otherwise

Thus, a formula may be satisfied, not satisfied, or left undetermined by an assignment s in $\mathfrak{A}'$.

Notice that for atomic formulas, the definition of satisfaction given above can be translated to :

$\mathfrak{A}' \models_T [f_i(x_0,\ldots,x_{n-1}) \geq r][s]$ iff $f_i^{\mathfrak{A}'}(s(x_0),\ldots,s(x_{n-1}))$, is defined and $\geq r$;

$\mathfrak{A} \models_T R_j\, x_0,\ldots,x_{n-1}\,[s]$ iff $R_j^{\mathfrak{A}'}(s(x_0),\ldots,s(x_{n-1}))$ is defined and equal to 1.

A partial $\mathbb{R}$-structure $\mathfrak{A}'$ represents what we actually know, or, at least, accept and are not willing to change. In MDC, the theoretical structures $\mathfrak{B}$ compatible with $\mathfrak{A}'$ (i.e. that are possible given $\mathfrak{A}'$) are the complete extensions of $\mathfrak{A}'$. Here, the situation will be different. There may be compatible theoretical structures which are not extensions of $\mathfrak{A}'$.

Now we are ready to relate $\mathfrak{A}$ to $\mathfrak{A}'$. We say that $\mathfrak{A}'$ is incompatible with the total $\mathbb{R}$-structure $\mathfrak{A}$ (given $\mathfrak{A}_e$), iff there is a formula ϕ and an assignment s in $\mathfrak{A}$ such that,

(i) $\mathfrak{A}' \models_T \phi[s]$,

(ii) $P_{\mathfrak{A}_e}(\phi, s)$ is low,

and

(iii) $P_{\mathfrak{B}_e}(\phi, s)$ is high, for some alternative to $\mathfrak{A}, \mathfrak{B}$.

In the account without considering random errors of MDC, $\mathfrak{A}$ was to be rejected, if it was not an extension of $\mathfrak{A}$. That is, if a sentence true in $\mathfrak{A}$, was false in $\mathfrak{A}'$ (or, more precisely, if there is a formula ϕ and an assignment s such that $\mathfrak{A} \models \phi[s]$, but $\mathfrak{A}' \models_T \neg \phi[s]$). In our present account, $\mathfrak{A}$ might not be an extension of $\mathfrak{A}'$, but anyway compatible with it, if $P_{\mathfrak{A}_e}(\phi, s)$ is high for all ϕ and s with $\mathfrak{A}' \models_T \phi[s]$. That is, everything that is approximately true in $\mathfrak{A}$ (i.e. has high probability in $\mathfrak{A}_e$) is true in $\mathfrak{A}'$, and there is nothing true in $\mathfrak{A}'$ that is approximately false in $\mathfrak{A}$ (i.e. has low probability in $\mathfrak{A}_e$).

How low the probabilities should be to reject $\mathfrak{A}$, depends fundamentally on the alternatives available. If there is a "reasonable alternative that assigns high probabilities to all sentences true in $\mathfrak{A}'$, then we might reject $\mathfrak{A}$, even though the probabilities in $\mathfrak{A}_e$ might not be very low. With no reasonable alternative, we would need very low probabilities, in order to reject $\mathfrak{A}$. The following is a possible explanation of what a reasonable alternative is. First, a definition. We say that the theory τ(in the similarity type τ) is confirmed by the B-τ-partial structure $\mathfrak{B}'$ (given $\mathfrak{A}_e$) if the total $\mathbb{R}$-structure $\mathfrak{B}$ determined by T for the objects in B, has the property that for all formulas ϕ and assignments s in B, if $\mathfrak{B}' \models_T \phi[s]$ then $P_{\mathfrak{B}_e}(\phi, s)$ is high. Suppose that if B is a set of objects that has been studied in a science, then $\mathfrak{B}_B$ is the B-τ-partial structure that is accepted as true, and assume that T is confirmed by all such $\mathfrak{B}_B$. Then, if Σ is the total $\mathbb{R}$-structure determined by such a T for the objects in A, it is a <u>reasonable alternative</u> to $\mathfrak{A}$.

The account given up to now is unrealistic in that it assumes that we measure each object just once. We could solve this problem by having several measurements, but assign one value to $f_i^{\mathfrak{A}'}(a_o, \ldots, a_{n-1})$, namely, their average. However, by using this procedure we lose some of the statistical power that may be available. In particular, with just one value assigned, we have no real hope of getting rid of $\mathfrak{A}_e$ in the definition of incompatibility. As given, we defined $\mathfrak{A}$ incompatible with $\mathfrak{A}'$ (given $\mathfrak{A}_e$). The standard deviations included in $\mathfrak{A}_e$ are not, usually, important for scientific theories.

In order to include repetitions of measurements, we introduce, for each $\mathfrak{A}$ and $\mathfrak{A}_e$, the structure $\mathfrak{A}_e^\omega$, called an ω-E-structure, with a language for this structure and a definition of probability for its formulas.

To all operations and relations in $\mathfrak{A}$, we add one more place to range over ω, the natural numbers; $f_i^{\mathfrak{A}_e^\omega}(a_o,\ldots,a_{n-1},t)$ is a random variable for each $a_o,\ldots,a_{n-1}\in A$ and $t\in\omega$, with the same distribution as $f_i^{\mathfrak{A}_e}(a_o,\ldots,a_{n-1})$; similarly, with relations, $R_j^{\mathfrak{A}_e^\omega}(a_o,\ldots,a_{n-1},t) = R_j^{\mathfrak{A}}(a_o,\ldots a_{n-1})$ for all $a_o,\ldots,a_{n-1},\in A$, $t\in\omega$.

The language is now a two-sorted language with variables $x,y,z\ldots$ for elements of A, and m,n for elements of ω. The atomic formulas are:

$x = y$

$m = n$

$R_j\, x_o,\ldots,x_{n_j-1},m$, for each $j\in J$

$[f_i(x_o,\ldots x_{n_i-1},m) \geq r]$, for each $r\in\mathbb{Q}$, $i\in I$.

This language will be a two-sorted $L_{\omega_1\omega}$-language with finitely many quantifiers for both types of variables. The assignments s,now, adscribe elements of A for the variables $x,y,z\ldots$, and elements of ω for the the other sort. Just as for $\mathfrak{A}_e$, $\mathfrak{A}_e^\omega$ assigns probabilities to formulas.

Let $\mathcal{B}_i^{\mathfrak{A}_e^\omega}(a_o,\ldots,a_{n-1})$ be the product algebra of $\mathcal{B}_i^{\mathfrak{A}_e}(a_o,\ldots a_{n-1})$ ω-times, and $\mu_i^{\mathfrak{A}_e^\omega}(a_o,\ldots,a_{n-1})$ its product measure. Then,

$$\mathcal{B}_i^{\mathfrak{A}_e^\omega} = \Pi < \mathcal{B}_i^{\mathfrak{A}_e^\omega}(a_o,\ldots,a_{n-1}) : a_o,\ldots,a_{n-1}\in A >$$

and $\mu_i^{\mathfrak{A}_e^\omega}$ is its corresponding product measure. For $j\in J$, $\mathcal{B}_j^{\mathfrak{A}_e^\omega}$ and $\mu_j^{\mathfrak{A}_e^\omega}$ are defined analogously. Finally, let

$$\mathcal{B}^{\mathfrak{A}_e^\omega} = \Pi < \mathcal{B}_k^{\mathfrak{A}_e^\omega} : k\in I\cup J >,$$

and let $\mu^{\mathfrak{A}_e^\omega}$ be its corresponding product measure.

An element $\mu\in\mathcal{B}^{\mathfrak{A}_e^\omega}$ is a system

$$\mu = < \mu_k : k\in I\cup J >.$$

If $k\in J$, then $\mu_k = < \mu_k(t) : t\in\omega >$ where $\mu_k(t)\in\mathcal{B}_k^{\mathfrak{A}_e}$. If $i\in I$, then

$\mu_i = < \mu_i(a_o,\ldots,a_{n-1},t) : a_o,\ldots,a_{n-1}\in A, t\in\omega >$, where $\mu_i(a_o,\ldots,a_{n-1},t) \in B_i^{\mathfrak{A}_e}(a_o\ldots,a_{n-1})$ for each $t\in\omega$.

h, now, assigns to each formula ϕ and assignment s of the new language an element of $B^{\mathfrak{A}_e^\omega}$, as follows:

(i) $$h(x=y;s) = \begin{cases} \mathbb{1}, & \text{if } s(x) = s(y) \\ \mathbb{0}, & \text{otherwise.} \end{cases}$$

$$h(n = m;s) = \begin{cases} \mathbb{1}, & \text{if } s(n) = s(m) \\ \mathbb{0}, & \text{otherwise.} \end{cases}$$

(ii) $h(R_j\ x_o,\ldots,x_{n-1},m;s) = \mu$ where

$$\mu_j(s(m)) = \begin{cases} 1_j, & \text{if } R_j^{\mathfrak{A}}(s(x_o),\ldots,s(x_{n-1})) = 1 \\ \mathbb{0}_j, & \text{otherwise,} \end{cases}$$

and $\mu_k(t) = 1_k$, for all $k\in I\cup J$, $t\in\omega$, with $k\neq j$ or $t\neq s(m)$.

(iii) $h([f_i(x_o,\ldots,x_{n-1},m) \geq r];s) = \mu$ where

$$\mu_i(s(x_o),\ldots, s(x_{n-1}),s(m)) = [f_i(s(x_o),\ldots,s(x_{n-1})) \geq r]$$

and $\mu_k(a_o,\ldots,a_{n-1}t) = 1_k(a_o,\ldots,a_{n-1})$, for all $k\neq i$ or

$$(a_o,\ldots,a_{n-1},t) \neq (s(x_o),\ldots,s(x_{n-1}),s(m))$$

(iv), (v), (vii), and (viii) are the same as before.

We need two more clauses:

(ix) $$h(\exists n\phi;s) = \bigvee_{t\in\omega} h(\phi,s_t^n).$$

(x) $$h(\forall n\phi;s) = \bigwedge_{t\in\omega} h(\phi,s_t^n).$$

Just as before, the probability in $\mathfrak{A}_e^\omega$ is given by:

$$P_{\mathfrak{A}_e^\omega}(\phi,s) = \mu^{\mathfrak{A}_e^\omega}(h(\phi;s)).$$

Now, the A,t- partial ω-structures (or partial structures with repetition), $\bar{\mathfrak{A}}$ are of the form

$$\bar{\mathfrak{A}} = \langle A, f_i^{\bar{\mathfrak{A}}}, R_j^{\bar{\mathfrak{A}}} \rangle_{i \in I,\ j \in J},$$

where $f_i^{\bar{\mathfrak{A}}}$ is a partial operation defined on ${}^{n_i}A \times \omega$ into $\mathbb{R}$ and $R_j^{\bar{\mathfrak{A}}}$ is a partial relation on ${}^{n_j}A \times \omega$. (Here, ${}^{n}A$ is the set of n-tuples of A). These functions may be partially defined on A, ω or both; e.g. $f_i^{\bar{\mathfrak{A}}}(a_o, \ldots, a_{n-1}, t)$ may be defined only for some $a_o, \ldots, a_{n-1} \in A$ and $t \in \omega$. In general, if we assume that $\bar{\mathfrak{A}}$ represents our actual knowledge, then, for each $a_o, \ldots, a_{n-1} \in A$ there will only be finitely many $t \in \omega$ with $f_i^{\bar{\mathfrak{A}}}(a_o, \ldots, a_{n-1} t)$ defined.

A complete extension $\bar{\mathfrak{B}}$ of $\bar{\mathfrak{A}}$ will have these functions defined everywhere in A and ω, and extend those of $\bar{\mathfrak{A}}$. Observe that in $\bar{\mathfrak{A}}$, or in any of its extensions $\bar{\mathfrak{B}}$, we may have $f_i^{\bar{\mathfrak{A}}}(a_o, \ldots, a_{n-1}, t) \neq f_i^{\bar{\mathfrak{A}}}(a_o, \ldots, a_{n-1}, v)$, for $t, v \in \omega$ with $t \neq v$.

Satisfaction for $\bar{\mathfrak{A}}$ is defined just as for the partial structures without repetitions $\mathfrak{A}'$.

In the language that we have introduced there is a formula ϕ and an assignment s such that

$$\bar{\mathfrak{A}} \models_T \phi[s] \quad \text{iff} \quad |t_i(a_o, \ldots, a_{n-1})| < a$$

where $t_i(a_o, \ldots, a_{n-1})$ is Student's t for the measurement $f_i^{\bar{\mathfrak{A}}}(a_o, \ldots, a_{n-1}, v)$ with $v \in \omega$ that are defined in $\bar{\mathfrak{A}}$, and $a \in \mathbb{R}$.

That is

$$t_i(a_o, \ldots, a_{n-1}) = \frac{M - f_i^{\mathfrak{A}}(a_o, \ldots, a_{n-1})}{S_M},$$

where M is the average of the sequence

$\langle f_i^{\bar{\mathfrak{A}}}(a_o, \ldots, a_{n-1}, v) : v \in \omega$ and $f_i^{\bar{\mathfrak{A}}}(a_o, \ldots, a_{n-1} v)$ is defined in $\bar{\mathfrak{A}} \rangle$ and S_M is its sample standard deviation.

As we mentioned in Section 2, there is an $a \in \mathbb{R}$, such that $P_{\mathfrak{A}_e}^{\omega}(\phi, s)$

is low for all ω-E-structures $\mathfrak{A}_e^\omega$, associated with $\mathfrak{A}$. Thus, the following definition makes sense.

We say that the partial ω-structure $\bar{\mathfrak{A}}$ is incompatible with $\mathfrak{A}$ iff there is a formula ϕ and an assignment s such that,

(i) $\bar{\mathfrak{A}} \models_T \phi[s]$,

(ii) $P_{\mathfrak{A}_e^\omega}(\phi,s)$ is low, for every ω-E-structure $\mathfrak{A}_e^\omega$ associated with $\mathfrak{A}$.

(iii) $P_{\mathfrak{B}_e^\omega}(\phi,s)$ is high, for a certain alternative to $\mathfrak{A}$, $\mathfrak{B}$, and a certain ω-E- structure $\mathfrak{B}_e^\omega$ associated with $\mathfrak{B}$.

If a certain ω-E-structure $\mathfrak{A}_e^\omega$ is preferred, because of theoretical reasons, over all other ω-E-structures associated with $\mathfrak{A}$, then we might relativize the definition of compatibility to this $\mathfrak{A}_e^\omega$, by changing (ii) to (ii)' : $P_{\mathfrak{A}_e^\omega}(\phi,s)$ is low.

However, the definition given (with (ii) instead of (ii)') is preferable, because it is independent of inessential theoretical features, such as standard deviations.

It can be shown, by arguments similar to those presented in Chuaqui 198+, that the statistical tests for hypothesis are a special case of these definitions for the situation of this paper. In particular, we can explain, in this fashion the approximation to falsehood by a sequence of probabilities decreasing to zero.

Two possible extensions of the models discussed here may be mentioned. In the first place, $f_i^{\mathfrak{A}_e}(a_o,\ldots,a_{n-1})$ may have a different distribution than the normal one. This may happen with some methods of measurement. The second possible extension is to non-deterministic theories. In this case, the theoretical structure $\mathfrak{A}$ itself may have random variables, i.e. $f_i^{\mathfrak{A}}(a_o,\ldots,a_{n-1})$ may itself be a random variable. This is a possible line of inquiring that we have not yet pursued.

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