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FACTUAL PROBABILITY AND BROWNIAN MOTION

by

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I. Introduction

In the framework of a factual definition of probability-presented originally by Chuaqui in [2] , [3] and modified in [4] - a formulation of the Brownian Motion process was given in [1]. That formulation, which used some non-standard concepts and techniques, has the advantage of considering Brownian Motion as a "fast" random walk. Nevertheless, a formal translation of that formulation to "classical" terms may appear rather obscure for those who have never worked with these techniques.

In this paper I adopt a quite different and classical point of view in order to formulate Brownian motion in the general framework of [3] where causal structures are introduced for the study of compound random phenomena.

Our purpose is to present a model which determines a probability measure, more precisely, a non probabilistic structure which gives rise to a probability space and a Brownian Motion defined on it. In this sense our problem consists in the non probabilistic representation of a probability space.

We will consider only a one dimensional Brownian motion. A generalization to more dimensions should not be difficult. Let us first define a Brownian motion.

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a) A Brownian motion is a stochastic process $(Z_t)_{0 \leq t \leq 1}$ defined on a probability space $(\Omega, \underline{A}, P)$ with independent increments, i.e. for every choice of parameters $t_1 < t_2 < \ldots < t_n$, the increments $Z_{t_2} - Z_{t_1}$, $Z_{t_3} - Z_{t_2}$, ..., $Z_{t_n} - Z_{t_{n-1}}$ are independent,

b) $Z_0 = 0$ $a.s.$,

c) If $0 \leq s < t$, the random variable $Z_t - Z_s$ is normally distributed with expectation 0 and variance $\sigma^2 (t-s)$ (σ a fixed positive number), i.e.

$$P(Z_t - Z_s < x) = \frac{1}{\sigma \sqrt{2\pi (t-s)}} \int_{-\infty}^{x} \exp\left(- \frac{\alpha^2}{2\sigma^2 (t-s)}\right) d\alpha$$

Usually the condition of the a.s. continuity of sample functions (i.e. the real functions $Z_t(w)$ of t) is required [8]. In this formulation of Brownian motion-as it was originally studied[6] - we do not require this condition. Nevertheless we will show afterwards that it is possible to construct a continuous version of this process.

We follow the notation and definitions in [3]. Some changes were introduced in [4] but they are not important for our purposes.

II. Simple Probability Structures and Normal Probability Law.

We intend to find a representation of the normal distribution on $(\mathbb{R}, \underline{B})$ in the framework of Chuaqui's simple probability models.

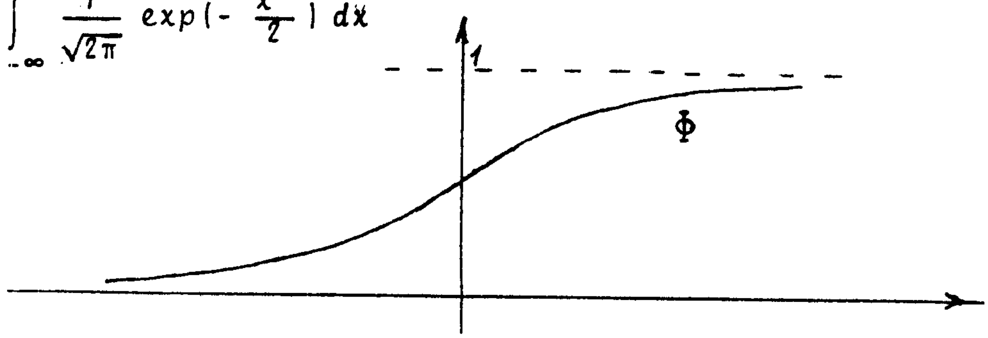
As a motivation we notice that the uniform probability distribution on $(0,1)$ is the probability measure defined on $((0,1), \underline{B}_{(0,1)})$ which is invariant under translations (the Lebesgue measure):

$$P(x \dotplus E) = P(E) \quad , \forall\, x \in (0,1) \quad , \forall\, E \in \underline{B}_{(0,1)}$$

($\dotplus$: addition modulo 1).

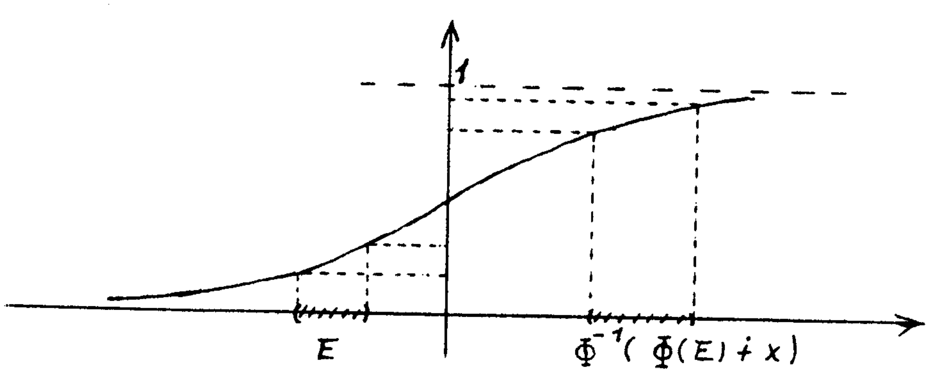
The probability measure induced by the normal law is also invariant under certain transformations: let Φ be the function defined by

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$



The probability P arising from the normal law is such that

$$P(\Phi^{-1}(x \dotplus \Phi(E))) = P(E) \quad , \quad \forall\, x \in (0,1) \quad , \quad \forall\, E \in \underline{B} \; .$$



We shall call the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(y) := \Phi^{-1}(x \dot{+} \Phi(y))$ a Φ - translation(in x) of the real line.

Let us consider the simple probability structure $\underline{\underline{K}} = \langle \underline{K}, \mathcal{L} \rangle$ where $\underline{K} = \{\mathcal{a}_r : r \in \mathbb{R}\}$, $\mathcal{a}_r = \langle \mathbb{R}, (I)_{I \in \mathcal{J}}, R, \{r\} \rangle$, $\mathcal{J}$ is the familly of intervals of the real line, $\mathcal{L} = \langle \mathbb{R}, (I)_{I \in \mathcal{J}} \rangle$ is the intrinsic part and R is the binary operation such that $Rab = \Phi^{-1}(\Phi(a) \dot{+} \Phi(b))$ and may be thought as the Φ - translation of a in $\Phi(b)$ (as Φ is invertible from $(0,1)$ into $\mathbb{R}$ we can represent each Φ - translation in this way).

According with the definitions above, we have that the structural part $\mathcal{a}_{r,st}$ of each $\mathcal{a}_r \in \underline{K}$ is $\mathcal{a}_{r,st} = \langle \mathbb{R}, R, \{r\} \rangle$.

$\underline{K}_{st} := \{\mathcal{a}_{r,st} : r \in \mathbb{R}\}$ determines the group $G_{\underline{K}}$ of permutations of $\mathbb{R}$ under which the probability measure has to be invariant; $\langle \mathbb{R}, (I)_{I \in \mathcal{J}} \rangle$ determines the σ - algebra $B_{\underline{K}}$ of events. In this case, $B_{\underline{K}}$ contains all Borelian subsets of $\mathbb{R}$.

We are interested in the group of permutations $G_{\underline{K}}$ since, according with the general definition, for each $B_1, B_2 \in B_{\underline{K}}$ one has $B_1 \sim B_2$ (B_1 symmetric with B_2) iff there exists some $f \in G_{\underline{K}}$ such that $B_2 = B_1^{f} = \{\mathcal{a} \in \underline{K} : \text{exists } \mathcal{a}' \in B_1 \text{ with } f^*(\mathcal{a}'_{st}) = \mathcal{a}_{st}\}$, and the probability measure μ determined by $\underline{\underline{K}}$ has to be such that, for $B_1 \sim B_2$, $\mu(B_1) = \mu(B_2)$.

For a permutation f of $\mathbb{R}$ to be in $G_{\underline{K}}$ it is necessary that $f^*(\mathcal{a}_{r,st}) = \mathcal{a}_{t,st}$ for some $t \in \mathbb{R}$ if $r \in \mathbb{R}$, *i.e.* $\langle \mathbb{R}, f^* R, \{fr\} \rangle = \langle \mathbb{R}, R, \{t\} \rangle$. Since $f^* \langle \mathbb{R}, R \rangle = \langle \mathbb{R}, R \rangle$, f has to be a Φ - translation of the real line, and therefore the probability measure μ agrees with that provided by the normal law, that is, $\mu(B') = \int_B \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})dx$, where

$B' = \{\mathfrak{a}_r \in \underline{K} : r \in B\}$, and B is a Borelian subset of $\mathbb{R}$.

In order to represent the probability measure corresponding to the normal law $N(\mu, \sigma^2)$, it suffices to replace the operation $R = R_\Phi$ by R_F where F is the function defined by

$$F(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{z} exp\left(-\frac{(x-\mu)^2}{\sigma^2}\right) dx \quad .$$

III. Causality and Brownian motion.

Let $\mathbb{T}$ be the closed unit interval $[0,1]$;

$$\ddot{\mathbb{T}} = \{S \cup \{0\} : S \subseteq \mathbb{T} \text{ and } 1 < |S - \{0\}| < \aleph_0\},$$

the family of finite subsets of $[0,1]$ which contains 0; and let R denote the usual order relation $<$ in $[0,1]$. Thus, our compound causal structure is $\mathbb{T} = \langle \mathbb{T}, \ddot{\mathbb{T}}, R \rangle$ (cf. [3]). If $X \in \ddot{\mathbb{T}}$, then $\langle X, R \rangle$ is a finite causal tree.

Now, we have to define our compound probability structure. In order to do this we will define the set $\mathbb{H}$ of outcomes and associate to each $t \in [0,1]$ a family of simple probability structures. The union of the members of this family may be interpreted as the totality of possible simple outcomes at time t (cf. [2], [3], [4]).

Associate to $t_0 := 0$ the simple probability structure $\underline{K}_0 = \{(\{0\},\{0\})\}$. In this structure we have probability $p_{t_0,0} = 1$.

1. Definitions: The function f is an outcome, that is, $f \in \mathbb{H}$, iff

i) there exists $X \in \ddot{\mathbb{T}}$ such that $Dom\ f = X$ and $f(0) \in \underline{K}_0$

ii) if $Dom\ f = X$ and $X = \{t_0, t_1, \ldots, t_n\}$ with $0 = t_0 < t_1 < \ldots < t_n$, then for each $k = 1, \ldots, n-1$:

$f(t_{k+1}) \in \underline{K}^r_{k+1}$, for some $r \in \mathbb{R}$ (which may be different for each k); where $\underline{K}^r_{k+1}$ is the simple probability structure $\underline{K}^r_{k+1} := \{(\mathbb{R}, \{s\}) : s \in \mathbb{R}\}$.

($(\mathbb{R}, \{s\})$ is only an abbreviation for $\langle \mathbb{R}, (I)_{I \in \mathcal{J}}, R_F, \{s\} \rangle$ where R_F is the operation in section II corresponding to the normal distribution function F)

Then the probability $P_{t_{k+1}, r}$ on $\underline{K}_{k+1}^{r}$ is given by

$$P_{t_{k+1}, r}(E) = \frac{1}{\sqrt{2\pi(t_{k+1} - t_k)}} \int_E \exp\left(-\frac{1}{2}\frac{(x-r)^2}{t_{k+1} - t_k}\right) dx$$

$$\left(\text{let } p_{t_{k+1}, r}(x) = \frac{1}{\sqrt{2\pi(t_{k+1} - t_k)}} \exp\left(-\frac{1}{2}\frac{(x-r)^2}{t_{k+1} - t_k}\right)\right.$$

be the density function).
where $E \in B_1$ is a Borel subset of the line (we assume $\sigma = 1$).
That is, the probabilities on $\underline{K}_{k+1}^{r}$ are distributed according to a normal probability law with mean r and variance $t_{k+1} - t_k$.
At t_1, $\delta(t_1) \in \underline{K}_1^0$, the simple probability structure $\underline{K}_1^0 := \{(\mathbb{R}, \{s\}) : s \in \mathbb{R}\}$ where the probability $P_{t_1,0}$ on $\underline{K}_1^0$ is

$$P_{t_1,0}(E) = \frac{1}{\sqrt{2\pi t_1}} \int_E \exp\left(-\frac{1}{2}\frac{x^2}{t_1}\right) dx \quad , \quad E \in B_1$$

(let $p_{t_1,0}(x) := \dfrac{1}{\sqrt{2\pi t_1}} \exp\left(-\dfrac{1}{2}\dfrac{x^2}{t_1}\right)$ be the density function)

(the Brownian particle starts from the origin).

iii) $\delta(t_k) = (\mathbb{R}, \{s\})$ implies

$$\delta(t_{k+1}) \in \underline{K}_{k+1}^{s} \quad , \quad k = 0, \ldots, n-1.$$

2. <u>Remarks</u>:

1) Though the mathematical problem of representation of the normal probability law through the simple probability structures is solved in section II a characterization of this law from factual considerations, e.g. symmetries, would be very interesting.

2) Clearly $\underline{\mathbb{H}} := < \mathbb{T}, \mathbb{H} >$ is a compound probability structure (see [3]). We can, for example, show the following: if $f \in \mathbb{H}$ and $Dom f = T$ (, *i.e.* if $f \in \mathbb{H}_T$), then $\mathbb{H}(f, t) := \{g(t) : g \in \mathbb{H}_T$ and $g \upharpoonright T_t = f \upharpoonright T_t\}$ is a simple probability structure. In fact, suppose $T = \{t_0, t_1, \ldots, t_n\}$ and $t = t_{k+1}$, then $T_t = \{t_0, \ldots, t_k\}$ and $\mathbb{H}(f, t_{k+1}) = \{g(t_{k+1}) : g \in \mathbb{H}_T$ and $g \upharpoonright \{t_0, \ldots, t_k\} = f \upharpoonright \{t_0, \ldots, t_k\}\} = \underline{K}_{k+1}^{f(t_k)}$.

3) The functions $p_{k,r}(\cdot)$ are the transition probability functions and play a similar role to that of transition probabilities in discrete Markov chains (see [1]). The basic idea is the following: at $t = t_{k+1}$ and in $\underline{K}_{k+1}^{r}$ we have all possible simple outcomes (positions at time t_{k+1}) given that at time t_k we had position r. Then the conditional mean value at t_{k+1} is r and the corresponding conditional variance is the time $t_{k+1} - t_k$ between $k^{\underline{th}}$ and $(k+1)^{\underline{th}}$ steps.

4) $\underline{\mathbb{H}}$ determines a measure μ on subsets of $\mathbb{H}$, the set of compound outcomes. In order to have "natural" outcomes we could redefine compound outcomes as subsets A of $\mathbb{H}$ with the following properties: (a) if $g, f \in \mathbb{H}$, then $g \cup f$ is a function (compatibility condition), and (b) A is maximal with respect to (a). In this case, only few changes in the definition of compound outcomes in [3] should be necessary.

We shall see in the following how Brownian motion appears.

3. <u>Definition</u>:

Let $< X, R >$ be any of our finite causal trees, say

$X = \{t_0, t_1, \ldots, t_n\}$ with $t_0 < t_1 < \ldots < t_n$.

We define a function F_X from $\mathbb{R}^n$ into $\mathbb{R}$

by $F_X(x_1, \ldots, x_n) := \mu\{f \in \mathbb{H}_X : f(t_1) < x_1, \ldots, f(t_n) < x_n\}$

4. **Remarks:**

1) To be precise, we should write var. $f(t_i) < x_i$ instead of $f(t_i) < x_i$, where var. $f(t_i)$ is the variable part of the model in $\underline{K}_i^{\text{var.} f(t_i)}$ which equals $f(t_i)$, i.e. var. $f(t_i) = s$ if $f(t_i) = (\mathbb{R}, \{s\})$. Nevertheless we make the identification.

2) Sometimes we write $F_{(t_1, \ldots, t_n)}(x_1, \ldots, x_n)$ instead of $F_X(x_1, \ldots, x_1)$

to make explicit the dependence on the parameters $t_1, \ldots, t_n$.

5. **Theorem:**

If $< X, R >$ is a finite tree, say $X = \{t_0, t_1, \ldots, t_n\}$ with $t_0 < t_1 < \ldots < t_n$, then

$$F_X(x_1, \ldots, x_n) = \int_{-\infty}^{x_1} \ldots \int_{-\infty}^{x_n} p_{t_1, 0}(u_1) p_{t_2, u_1}(u_2) \ldots$$

$$\ldots p_{t_n, u_{n-1}}(u_n) du_1 \ldots du_n$$

Proof: we prove the theorem for $n = 2$; the general case may be obtained by induction on n. $X = \{t_0, t_1, t_2\}$.

$F_{(t_1, t_2)}(x_1, x_2) = \mu\{f \in \mathbb{H} : \mathit{Dom}\, f = X \text{ and } f(t_1) < x_1, f(t_2) < x_2\}$

We denote by A the event on the right side.

The measure μ on $\mathbb{H}$ is defined by induction on the ordinals. We recall some definitions from [3] : for a tree $< T,R >$ and $t \in T$, $T_t := \{s \in T : sRt$ and $s \neq t\}$, $\bar{T}_t := \{s \in T : sRt\}$, T'_α is the set of all minimal elements of $T - \cup\{T'_\beta : \beta \subset \alpha\}, T_\alpha = \cup\{T'_\beta : \beta \subset \alpha\}$, $\bar{T}_\alpha = \cup\{T'_\beta : \beta \subseteq \alpha\}$, $A(S) = \{f \upharpoonright S : f \in A\}$ with $A \subseteq \mathbb{H}_T$ and $S \subseteq T$.

In our case, $A \subseteq \mathbb{H}_X(\bar{X}_{t_2}) = \{f \upharpoonright \bar{X}_{t_2} : f \in \mathbb{H}$ and $Dom\, f = X\}$,

$\bar{X}_{t_2} = X$, $X'_i = \{t_i\}$, $X_0 = \phi$, $X_1 = \{t_0\} = \bar{X}_0$,

$$X_2 = \{t_0, t_1\} = \bar{X}_1 \ , \quad X_3 = \{t_0, t_1, t_2\} = \bar{X}_2$$

and t_2 $X'_2 = \{t_2\}$. The measure on $\mathbb{H}_X(\bar{X}_{t_2})$ is $\bar{\mu}_{t_2}$ and is given by $\bar{\mu}_{t_2}(A(\bar{X}_{t_2})) = \int_{A(X_{t_2})} \mu_{f,t_2}(A(f,t_2))d\mu_{t_2}$ (*)

where μ_{t_2} is measure given by

$\mu_{t_2} = \pi < \bar{\mu}_s : s \in X'_1 \ , \ s < t_2 > = \bar{\mu}_{t_1}$ (cf. [3]) and

μ_{f,t_2} is the measure in the simple probability structure $\mathbb{H}(f,t_2) = \underline{K}_2^{f(t_1)}$.

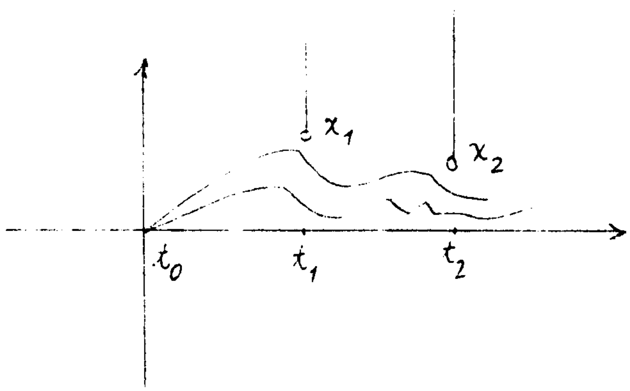
As $A(\bar{X}_{t_2}) = A$, from (*) we have $\mu(A) = \int_{\mathcal{E}} \mu_{f,t_2}(A(f,t_2))d\bar{\mu}_{t_1}$,

$\mathcal{E} := \{f \upharpoonright \{t_0,t_1\} : f(t_1) < x_1, f(t_2) < x_2 \ , \ f \in \mathbb{H}_X\}$, but

$A(f,t_2) = \{g(t_2) : g \in A \ , \ g \upharpoonright \{t_0,t_1\} = f \upharpoonright \{t_0,t_1\}\} =$

$\{g(t_2) : g \in A$ and $g(t_1) = f(t_1)\} = (-\infty, x_2) \subseteq K_2^{f(t_1)}$, then $\mu_{f,t_2}(A(f,t_2)) = \mu_{f,t_2}(-\infty, x_2) =$

$$\frac{1}{\sqrt{2\pi(t_2-t_1)}} \int_{-\infty}^{x_2} e^{-\frac{1}{2}\frac{(x-f(t_1))^2}{t_2-t_1}} dx$$



$$\mu(A) = \int_{\mathcal{C}} \int_{-\infty}^{x_2} \frac{1}{\sqrt{2\pi(t_2-t_1)}} e^{-\frac{1}{2}\frac{(x-f(t_1))^2}{t_2-t_1}} dx\, d\bar{\mu}_{t_1} \qquad (**)$$

Now, $\mu_{t_1} = \pi < \bar{\mu}_s : s \in X_0' \quad s < t_1 >$

$= \bar{\mu}_{t_0} = \mu_{f,t_0}$

$= 1$; (the measure in $\mathbb{H}(f,t_0) = (\{0\},\{0\})$)

$\bar{\mu}_{t_1}$ is the probability until t_1 (included t_1).

If suffices to show that $\bar{\mu}_{t_1}$ coincides with the probability in $\underline{K}_1^0$, i.e. it is given through the transition density $p_{t_1,0}$: In fact,

$$\bar{\mu}_{t_1}(B) = \int_{B(X_{t_1})} \mu_{f,t_1}(B(f,t_1))\, d\mu_{t_1} \qquad (***)$$

for any $B \subseteq \mathbb{H}_{\bar{X}_{t_1}}$. But $B(X_{t_1}) = B(\{t_0\})$

$= \{f \equiv 0\}$ and $B(f,t_1) = \{g(t_1) : g(t_0) = f(t_0),\ g \in B\} = \mathbb{R}$,

then in (***) $\bar{\mu}_{t_1}(B) = \mu_{0,t_1}(B)$. $p_{t_1,0}$ is precisely the probability density function which defines

μ_{0,t_1}, then $d\,\mu_{0,t_1}(y) = p_{t_1,0}(y)\, dy$

$$= \frac{1}{\sqrt{2\pi t_1}} \exp\left(-\frac{1}{2}\frac{y^2}{t_1}\right) dy$$

and in (**), as $f(t_1)$ may be any real number in $(-\infty, x_1]$, we have

$$\mu(A) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} \frac{1}{\sqrt{2\pi(t_2-t_1)}}\, e^{-\frac{1}{2}\frac{(x-y)^2}{t_2-t_1}}\, dx\, \frac{1}{\sqrt{2\pi t_1}}\, e^{-\frac{1}{2}\frac{y^2}{t_1}}\, dy$$

$$= \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} p_{t_1,0}(y)\, p_{t_2,y}(x)\, dx\, dy$$

$$= \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} p_{t_1,0}(u_1)\, p_{t_2,u_1}(u_2)\, du_1 du_2 \quad ;$$

and the proof is complete.

6. <u>Corollary</u>:

For each X, F_X is a probability distribution function (in the usual sense) and determines probability measures P_X on $(\mathbb{R}^{|X|-1}, B_{|X|-1})$ in the natural way. (B_n denotes the family of Borelian sets in $\mathbb{R}^n$)

Proof: it may be shown directly that

$$\Phi_{(t_1,\ldots,t_n)}(x_1,\ldots,x_n) := \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} p_{t_1,0}(u_1) p_{t_2,u_1}(u_2) \cdots p_{t_n,u_{n-1}}(u_n) du_1 \ldots du_n$$

defines an n-dimensional probability distribution function. Once we have this distribution function Φ_X in $\mathbb{R}^n$, we may define a probability measure P on B_n by extending uniquely the (elementary) probability measure $P(B) := \Phi_X(x_1\ldots, x_n)$, where $B = (-\infty, x_1]\text{x}\ldots\text{x}(-\infty, x_n]$, to all Borelian subsets of $\mathbb{R}^n$ (see [7] chap. 1)

7. Remark:

We shall see now how Brownian motion appears in a very natural way in a canonical probability space $(\Omega, \underline{B}, P)$ where P is a unique "extension" of the measure μ on $\mathbb{H}$ to Ω through the distribution functions F_X.

8. Definition

$\Omega := \mathbb{R}^{\mathbb{I}}$, the set of functions from the unit interval $[0,1]$ into $\mathbb{R}$.
$\underline{B}$ is the σ - algebra generated by the cylinder sets, i.e. by the subsets of Ω of the form $\{\omega \in \Omega: (\omega(t_1),\ldots,\omega(t_k)) \in B\}$ with $B \in B_k$. Define random variables Z_t, $t \in \mathbb{I}$, to be the coordinate functions, i.e. $Z_t(\omega) = \omega(t)$.

9. Remark:

The probability functions P_X in corollary 6., or equivalently, the $P_{(t_1,\dots,t_n)}$'s, are defined for $t_0 < t_1 < \dots < t_n$. In order that the compability conditions

i) $P_{(t_{\pi_1},\dots,t_{\pi_n})}(S) = P_{(t_1,\dots,t_n)}(\pi^{-1}S)$

, $S \in B_n$, (here π denotes a permutation of $1, \dots, n$ and also the bijection from $\mathbb{R}^n$ into $\mathbb{R}^n$ defined by $\pi(x_1,\dots,x_n) = (x_{\pi_1},\dots,x_{\pi_n})$).

ii) $P_{(t_1,\dots,t_n)}(S) = P_{(t_1,\dots,t_{n+m})}(S \times \mathbb{R}^{m-n})$

are satisfied, we consider (i) as a definition of $P_{(t_1,\dots,t_n)}$, where the t_i's, which are different, need not increase with the subindex. The same holds for the distribution functions F_X.

10. Lemma:

The probability functions P_X satisfy the compatibility conditions in 9.

Proof: (i) holds by definition; for (ii) it suffices to show the compatibility conditions for the F_X's . Condition (ii) is satisfied by the Φ_X's in 6.

11. Lemma:

There is a unique probability measure P on $(\Omega, \underline{B})$ such that $P([(Z_{t_1},\dots,Z_{t_n}) \in S]) = P_{(t_1,\dots,t_n)}(S)$, $S \in B_n$

Proof: it suffices to follow the proof of the well-known theorem of Kolmogorov about the existence of a family of random variables on a common probability space corresponding to a family of finite dimensional distributions (see, e.g.[7] chap. 1).

The theorem, as it is usually presented, states only the existence of a common probability space, but the usual proof constructs the space precisely as we need it here.

12. Corollary:

The finite dimensional joint distribution functions of the random variables $(Z_t)_{t \in \mathbb{T}}$ defined in $(\Omega, \underline{B}, P)$ are given by
$P(Z_{t_1} \leqslant x_1, \ldots, Z_{t_n} \leqslant x_n) = F_{(t_1, \ldots, t_n)}(x_1, \ldots, x_n)$.

13. Theorem:

The random variables $(Z_t)_{t \in \mathbb{T}}$ defined in $(\Omega, \underline{B}, P)$ satisfy I(a) - (c), that is, the stochastic process $(Z_t)_{t \in \mathbb{T}}$ is a Brownian motion.

Proof: One has only to use the joint distribution functions

$$\Phi_{(t_1, \ldots, t_n)}(x_1 \ldots x_n) = \int_{-\infty}^{x_1} \ldots \int_{-\infty}^{x_n} p_{t_1, 0}(u_1) p_{t_2, u_1}(u_2) \ldots p_{t_n, u_{n-1}}(u_n) du_1 \ldots du_n$$

14. Remark:

Within this formulation it is not possible to obtain the a.s. continuity of trajectories. Nevertheless it may be obtained by considering a denumerable dense subset of $[0,1]$, showing that the restriction of Z_t to this subset is uniformly continous and extending Z_t to the whole of $[0,1]$ by continuity (cf. [5] for separable random processes).

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