Circumscription and Generic Mathematical Objects

Leopoldo E. Bertossi
Pontificia Universidad Catolica de Chile
Escuela de Ingenieria
Departamento de Ciencia de la Computacion
Casilla 306, Santiago 22, Chile
e-mail: bertossi@ing.puc.cl

Raymond Reiter
Department of Computer Science
University of Toronto
Toronto, Canada M5S 1A4
and
The Canadian Institute for Advanced Research
e-mail: reiter@ai.toronto.edu

Abstract

We investigate the possibility of using circumscription for characterizing the concept of a generic object in the context of a formalized mathematical theory. We show that conventional circumscriptive policies do not give the intuitively expected results for elementary geometry, and that there is a common explanation for this failure and the failure of circumscription in some standard instances of commonsense reasoning. It turns out, however, that scoped circumscription does provide the right mechanism for this task.

1 Generic Mathematical Objects

One of the motivations for the development of nonmonotonic logical formalisms was to characterize the concept of a generic or prototypical object as it arises in natural language and commonsense reasoning (see Reiter [13]). However, we also find this concept in mathematics. The best example is provided by set theory, more specifically in connection with the method of forcing, where the concept of generic set is a well defined notion in the theory. Generic sets help us prove certain propositions without using any of their "specific" properties (see [1]). Nevertheless, the concept of generic object also appears in its intuitive, non-formalized version in the context of a mathematical theory. In geometry, for example, a triangle without any special properties, e.g. right-angled or isosceles, etc. or lines without special positions with respect to a coordinate system, would be considered generic.

Some computational systems for mechanical theorem proving in geometry use auxiliary diagrams to guide the proofs of theorems. These diagrams are constructed with genericity in mind. As P.C. Gilmore puts it in his logical reconstruction of Gelernter's Geometry Machine [4]: "One more qualification is generally added in order not to admit as a diagram a set of points with special relationships that are not part of the premises ..." ([5, page 179]).

Our concern in this paper is with formalizing the concept of a generic mathematical object. Specifically, we explore the possibility of using circumscription (McCarthy [8, 9]) for this purpose. We will restrict ourselves to the case of elementary plane geometry. For this theory, we will show that conventional forms of circumscription are inadequate for defining the property of a generic triangle. We will show, on the other hand, that scoped circumscription (Etherington, Kraus and Perlis [3]) is exactly what is required.

2 Elementary Geometry and Generic Triangles

Let L be a first-order language for plane geometry. It is well known (see [21], [17] or [11]) that geometry can be finitely axiomatized starting with only two primitive predicates $\beta(.,.,.)$ and $\delta(.,.,.,.)$. Here, $\beta(x,y,z)$ means that point x lies between points y and z; $\delta(x,y,u,v)$ means that the distance between points x and y equals that between points u and v. Let G' be such a finite axiomatization, say, any of the equivalent axiomatizations $\mathcal{E}_2''$ in [21] or $\mathcal{P}^*$ in [17]. We can define in L the predicates O, S_1, S_2 for "triangle", "isosceles triangle", "right-angled triangle", respectively. Let us denote by D_O, D_1, D_2 the corresponding axioms. Then, D_O is

$$O(x,y,z) \equiv \neg(\beta(x,y,z) \vee \beta(y,z,x) \vee \beta(z,x,y)), \tag{1}$$

D_1 is

$$S_1(x,y,z) \equiv O(x,y,z) \wedge \delta(x,y,y,z),$$

and D_2 is

$$S_2(x,y,z) \equiv \tau(x,y,z) \vee \tau(y,x,z) \vee \tau(z,x,y),$$

where $\tau(x,y,z)$ means "(x,y,z) is a triangle with right angle at x" and can be defined by the following axiom [17]:

$$\begin{aligned} \tau(x,y,z) \equiv (\exists u,v)[&u \neq y \wedge u \neq z \wedge v \neq y \wedge v \neq z \wedge \\ &\delta(x,y,x,v) \wedge \delta(x,z,x,u) \wedge \delta(y,z,u,v) \wedge \delta(y,z,z,v) \wedge \\ &\delta(y,z,y,u)]. \end{aligned} \tag{2}$$

The objects of interest are the triangles. Singular triangles are the right-angled and isosceles triangles. Our focus is on the "generic" triangles, that is, in the triangles outside $S_1 \cup S_2$. Let G be the conjunction of G' with the definitions D_O, D_1, D_2, and let $\bar{a} = (a,b,c)$ be a triple of constants of L (corresponding to points) that do not appear in G'.

We would like $\bar{a}$ to be a "generic" triangle. The natural first approach is to circumscribe the singular properties of triangles, that is, to compute

$$Circum(G \wedge O(\bar{a}); S_1, S_2). \tag{3}$$

Unfortunately, this circumscription does not entail $\neg S_1(\bar{a}) \wedge \neg S_2(\bar{a})$, so that we cannot conclude that $\bar{a}$ is a generic triangle. In fact, models $\mathcal{M} = (M, \ldots, O^M, S_1^M, S_2^M, \bar{a}^M)$ of $G \wedge O(\bar{a})$ with $\bar{a}^M \in S_1^M$ (or S_2^M) are $<^{S_1,S_2}$-minimal. This is so since the theory contains definitions of the singular predicates, and the extensions in the model of these predicates are completely determined by the extension of M and the extensions of the basic predicates. In particular, if $\mathcal{M}'$ is a model of $G \wedge O(\bar{a})$ and $\mathcal{M}' \leq^{S_1,S_2} \mathcal{M}$, then $\mathcal{M}' = \mathcal{M}$. This can be seen syntactically as well since the following is valid:

$$Circum(G \wedge O(\bar{a}); S_1, S_2) \equiv G \wedge O(\bar{a}).$$

This circumscription does not prevent $\bar{a}$ from belonging to $S_1 \cup S_2$.

We have seen that there are $<^{S_1,S_2}$-minimal models where $\bar{a}$ is a right-angled triangle (or an isosceles triangle) and others where this is not the case. $\bar{a}$ is fixed in the circumscription (3) above. The possibility of allowing $\bar{a}$ to vary in the circumscription process (so that it can be accommodated outside the minimized S_i's), i.e. considering the circumscription

$$Circum(T; S_1, S_2; \bar{a}), \tag{4}$$

with variable $\bar{a}$, does not help either. It can be proved that the minimization in (4) does not depend on $\bar{a}$; the same argument preceding this remark applies in this case.

Notice that in the circumscriptions above the basic predicates β and δ are fixed, and this could be the reason for not obtaining the desired conclusion. The natural question is whether we can do better by allowing β and δ to vary in the circumscription. Since O is defined in terms of β and δ, we must allow this predicate to vary also. Therefore, we consider the circumscription:

$$Circum(G \wedge O(\bar{a}); S_1, S_2; \beta, \delta, O) \tag{5}$$

We now show that circumscription with variable predicates does not give the desired result either. We will use the predicate $\tau(x, y, z)$ whose intended meaning is "(x, y, z) determine a triangle with right angle at x". As we have seen, this predicate can be defined in terms of β and δ (formula (2)). Let D_τ be this axiom that defines τ. Instead of (5), we will consider the apparently simpler circumscription:

$$Circum(G' \wedge D_\tau \wedge D_O \wedge O(\bar{a}); \tau; \beta, \delta, O) \tag{6}$$

That is, our axioms consist of the usual axioms G' for geometry, the axiom defining a class τ of singular triangles, and the axiom D_O defining "trianglehood" (axiom (1)). By means of circumscription (6) we would like to conclude that (a, b, c) does not form a right-angled triangle, i.e. $\neg\tau(a, b, c)$, $\neg\tau(c, a, b)$ and $\neg\tau(b, c, a)$. This conclusion is impossible:

Theorem 1 $Circum(G' \wedge D_\tau \wedge D_O \wedge O(a,b,c); \tau; \beta, \delta, O) \not\models \neg\tau(a,b,c)$.

Proof: There is a model $\mathcal{M} = (M, \beta^M, \delta^M, \tau^M, O^M, a^M, b^M, c^M)$ of $G'' := G' \wedge D_\tau \wedge D_O \wedge O(a,b,c)$ with $(a^M, b^M, c^M) \in \tau^M$. We show that this model is $<^{\tau;\beta,\delta,O}$-minimal. More precisely, we prove there is no model $\mathcal{M}'$ of G'' with $\mathcal{M}' <^{\tau;\beta,\delta,O} \mathcal{M}$ where (a,b,c) is not a $\mathcal{M}'$-right-angled triangle (and then, $\neg\tau(a,b,c)$ is not true in all minimal models). So, assume to the contrary that there is such a model $\mathcal{M}' = (M', \beta^{M'}, \delta^{M'}, \tau^{M'}, O^{M'}, a^{M'}, b^{M'}, c^{M'})$ of G''. Then

1. $M' = M$,
2. $(a^{M'}, b^{M'}, c^{M'}) = (a^M, b^M, c^M)$,
3. $\tau^{M'} \subseteq \tau^M$,
4. $(a^{M'}, b^{M'}, c^{M'}) \notin \tau^{M'}$.

Since $\mathcal{M}'$ is a model of G'', (a^M, b^M, c^M) is a triangle in $\mathcal{M}'$ and there is a unique $\mathcal{M}'$-perpendicular to the $\mathcal{M}'$-segment $\overline{a^M b^M}$ through c^M. This perpendicular determines a point $d \in M$ in this segment for which $(d, b^M, c^M) \in \tau^{M'}$ holds. We may assume without loss of generality that $d \neq a^M, b^M, c^M$ (otherwise consider the perpendicular through b^M).

It is known (see [17]) that collinearity (i.e. $\neg O(x,y,z)$) can be defined in terms of τ by the axiom $\neg O(x,y,z) \equiv \exists r(\tau(r,x,y) \wedge \tau(r,x,z))$. [1] Since a^M, d, b^M are $\mathcal{M}'$-collin-ear, there is a $r \in M$ such that $(r, a^M, d) \in \tau^{M'}$ and $(r, a^M, b^M) \in \tau^{M'}$. Then, for this same r, $(r, a^M, d) \in \tau^M$ and $(r, a^M, b^M) \in \tau^M$. This is equivalent to a^M, d, b^M being $\mathcal{M}$-collinear. Then $(d, b^M, c^M) \notin \tau^M$ (because $d \neq a^M$ and a^M is already the intersection of the $\mathcal{M}$-perpendicular to the segment $\overline{a^M b^M}$ through c^M). Thus, $(d, b^M, c^M) \in \tau^{M'} \setminus \tau^M$, contradicting (3).

□

With similar arguments we can prove that circumscription (5) does not give the intuitively expected result either. Accordingly, none of the possible circumscriptions (with or without variable constants and predicates) leads to the intuitively desirable conclusions.

[1] Actually, it is possible to axiomatize geometry in terms of τ only, that is, both β and δ can be defined in terms of τ (see [18]).

3 The Lottery Paradox Revisited

In [3], Etherington, Kraus and Perlis survey various examples where circumscription (and other nonmonotonic formalisms) yield undesirable consequences. One such example is Kyburg's lottery paradox [6] (see also Poole's translation to ornithology [10]). As we shall now show, this paradox arises in the setting of elementary geometry, and thus provides an heuristic reason for the failure of circumscription to characterize genericity in geometry.

Abstractly, the lottery paradox for nonmonotonic reasoning arises whenever the axioms entail a disjunction $D_1 \vee \cdots \vee D_n$, and the default conclusions $\neg D_i$ are warranted for $i = 1, \ldots, n$. It is possible to explain the failure of circumscription in our geometric example in similar terms: elementary geometry entails the existence of a ternary function f defined by:

$$f(x, y, z) = \textit{ the projection of } x \textit{ on segment } \overline{yz}.$$

(the projection of x on $\overline{yz}$ is defined as x if x, y, z are collinear)

Using this function, we can represent, in disjunctive form, membership in the domain of triangles:

$$O(x, y, z) \equiv \tau(x, y, z) \vee (\tau(f(y, x, z), x, y) \wedge \tau(f(z, x, y), x, z)), \qquad (7)$$

that is, (x, y, z) forms a triangle if and only if (x, y, z) forms a triangle with right angle at x or it is possible to draw the projections of y and z on the opposite sides, obtaining two right-angled triangles.

Assume that we have the default rule

$$O(x, y, z) \xrightarrow{d} \neg\tau(x, y, z), \qquad (8)$$

that can be applied directly, in one step, in front of evidence of the form $O(x, y, z)$. If we have that $O(a, b, c)$, that is, (a, b, c) forms a triangle, we could conclude by (8), that $\neg\tau(a, b, c)$. Now, if $(f(b, a, c), a, b)$ or $(f(c, a, b), a, c)$ do not form a triangle, then the second disjunct in (7) is false. If both form a triangle, then, again by (8), we obtain that $\tau(f(b, a, c), a, b)$ and $\tau(f(c, a, b), a, c)$ are both false. In any case, the RHS in (7) becomes false, and we obtain a contradiction, namely, that (a, b, c) does not form a triangle.

Summarizing, predicate $O(x, y, z)$ can be defined by a disjunction. If $O(a, b, c)$ is given, we can apply default rule (8) to each of the disjuncts in the definition of $O(x, y, z)$, making all of them false, and thus obtaining a contradiction.

4 Scoped Circumscription and Generic Triangles

In this section we will see that the recently defined "scoped circumscription" [3] does allow us to obtain the desired conclusion, namely, that $\bar{a}$ is not a singular triangle.

Etherington, Kraus and Perlis [3] introduced scoped circumscription as a special case of the first order schema for formula circumscription [9]. We here rely on the second order version of their schema. $Circum_{Scope}(A; W(\bar{P}, \bar{x}))$, the *scoped circumscription* of formula $W(\bar{P}, \bar{x})$ in the theory A with variable predicate parameters $\bar{P}$ and $Scope$ is given by the second order sentence:

$$\begin{array}{l} A(\bar{P}', Scope') \wedge \forall \bar{x}[W(\bar{P}', \bar{x}) \wedge Scope'(\bar{x}) \supset W(\bar{P}, \bar{x}) \wedge Scope(\bar{x})] \\ \supset \forall \bar{x}[W(\bar{P}, \bar{x}) \wedge Scope(\bar{x}) \supset W(\bar{P}', \bar{x}) \wedge Scope'(\bar{x})]. \end{array} \tag{9}$$

With scoped circumscription we are introducing the scope of default reasoning by using a new predicate $Scope$. The scope of the domain of individuals we are reasoning about, more precisely, the restriction of predicate W to $Scope$, is kept narrow by minimization. Other individuals may remain outside the scope of reasoning. We see in scoped circumscription an interesting new technique for applying McCarthy's formula circumscription. This form of circumscription avoids some of the problems caused by more traditional and naive applications of circumscription, as discussed in [3].

Here, we will let G be elementary geometry with the definition (1) of predicate O for triangles. Let us introduce a new predicate $Scope$ to the language of elementary geometry. We want point $\bar{a}$ to be in the scope of our default reasoning. Nevertheless, we do not want all possible triangles to be in this scope.

In (9), we will take $A(\bar{P}, Scope)$ as the formula:

$$G \wedge \forall \bar{x}[\tau(\bar{x}) \equiv W((\beta, \delta), \bar{x})] \wedge O(\bar{a}) \wedge Scope(\bar{a}), \tag{10}$$

where $\bar{P}$ is (β, δ, O, τ), and $W((\beta, \delta), \bar{x})$ is the RHS of the axiom D_τ (formula (2)).

By taking A as the base theory (10), and $W((\beta, \delta), \bar{x})$ as the minimized formula (we could also take $\tau(\bar{x})$ by the second conjunct in A), we obtain from (9) the scoped circumscription corresponding to the predicate circumscription (6):

$$Circum_{Scope}(A; W((\beta, \delta), \bar{x})). \tag{11}$$

Our concern is whether this scoped circumscription gives the expected result.

Lemma 1 *Let A be a sentence of L and $Def := (\forall x)[\bar{P}(\bar{x}) \equiv E(\bar{Q}, \bar{x})]$ be an explicit definition of the tuple of predicates $\bar{P}$ with respect to A (so that $E(\bar{Q}, \bar{x}) \in L, \bar{P} \notin L$ and $\bar{Q} \in L$ is a tuple of predicates). Then the following is valid:*

$$Circum(A \wedge Def; \bar{P}; \bar{Q}) \equiv Circum(A; (\lambda \bar{x}) E(\bar{Q}, \bar{x}))$$

Proof: ($\Rightarrow$): $Circum(A \wedge Def, \bar{P}; \bar{Q})$ is defined by

$$\begin{aligned} A(\bar{Q}) \wedge \bar{P} = (\lambda \bar{x}) E(\bar{Q}, \bar{x}) \wedge (\forall \bar{X}, \bar{Z})[A(\bar{Z}) \wedge \\ \bar{X} = (\lambda \bar{x}) E(\bar{Z}, \bar{x}) \supset \neg(\bar{X} < \bar{P}))] \end{aligned}$$

and is equivalent to:

$$A(\bar{Q}) \wedge \bar{P} = (\lambda \bar{x}) E(\bar{Q}, \bar{x}) \wedge (\forall \bar{Z})[A(\bar{Z}) \supset \neg((\lambda \bar{x}) E(\bar{Z}, \bar{x}) < \bar{P})]. \tag{12}$$

(the abbreviations "$=$", "$<$" used in these sentences have a natural meaning when used with tuples of predicates; see [7] for definitions)

Sentence (12) implies:

$$\begin{aligned} A(\bar{Q}) \wedge (\forall \bar{Z})[A(\bar{Z}) \wedge (\lambda \bar{x}) E(\bar{Z}, \bar{x}) \leq (\lambda \bar{x}) E(\bar{Q}, \bar{x}) \\ \supset (\lambda \bar{x}) E(\bar{Z}, \bar{x}) = (\lambda \bar{x}) E(\bar{Q}, \bar{x})], \end{aligned} \tag{13}$$

which is precisely the definition of formula circumscription.

($\Leftarrow$): In this case we have to be more precise in our statement because the second circumscription is a sentence of L and the first is a sentence of the language expanded by the predicates $\bar{P}$. What we want to prove is that, for every model $(M, \bar{Q}^M)$ of the second circumscription, its expansion $(M, \bar{Q}^M, \bar{P}^M)$, with $\bar{P}^M$ defined by $\bar{P}^M := \{\bar{m} \in M | (M, \bar{Q}^M) \models E(\bar{Q})[\bar{m}]\}$, is a model of the first circumscription.

It is easy to check that with this natural interpretation of the implication, (13) implies (12).

□

For interesting remarks about the relationship between circumscription and definability see [2].

Corollary 1 *The scoped circumscription (11) (together with the base theory A) is logically equivalent to the second order predicate circumscription:*

$$Circum(A \wedge Def; \tau_{scoped}; \beta, \delta, O, \tau, Scope), \tag{14}$$

where Def is the axiom $\forall \bar{x}[\tau_{scoped}(\bar{x}) \equiv W((\beta,\delta),\bar{x}) \wedge Scope(\bar{x})]$ that defines the minimized new predicate τ_{scoped}. Predicates $\beta, \delta, O, \tau, Scope$ are variable in this predicate circumscription.

Proof: Use Lemma 1 with $E(\bar{Q},\bar{x})$ given by $W((\beta,\delta),\bar{x}) \wedge Scope(\bar{x})$, i.e. $\bar{Q}$ is $(\beta,\delta,Scope)$. $\Box$

As a consequence of this corollary, the question asked before is equivalent to asking whether the circumscription (14) entails $\neg\tau(\bar{a})$, that is, whether $\bar{a}$ is not a singular triangle. We would like this to be the case.

Notice that if we try to prove that circumscription (14) does not entail $\neg\tau(\bar{a})$ along the lines of the proof of Theorem 1, then the corresponding arguments do not apply. The reason is that, instead of 3. which we had there, we have now

$$\tau_{scoped}^{M'} \subseteq \tau_{scoped}^{M},$$

which is equivalent to

$$W^{M'} \cap Scope^{M'} \subseteq W^{M} \cap Scope^{M}.$$

In this form, the use of predicate $Scope$ in the scoped circumscription prevents us from applying our previous arguments to obtain the undesirable conclusion that $\bar{a}$ may be a singular triangle in minimal models. In fact, for the circumscription (14) of Corollary 1, we do obtain the conclusion that $\bar{a}$ is not a singular triangle.

Theorem 2

$$Circum(A \wedge Def; \tau_{scoped}; \beta, \delta\, O, \tau, Scope) \models \neg\tau(\bar{a}) \tag{15}$$

Proof: The circumscription in (15) is defined by:

$$\begin{array}{l} A(\bar{P}, Scope) \wedge Def(\bar{P}, Scope, \tau_{scoped}) \wedge \\ \quad (\forall \beta', \delta', O', \tau', X, Scope')[X < \tau_{scoped} \supset \\ \qquad \neg(A(\beta', \delta', O', \tau', Scope') \wedge Def(\tau', Scope', X))] \end{array} \tag{16}$$

Let $\mathcal{M} = (M, \beta^M, \delta^M, O^M, \tau^M, \tau^M_{scoped}, Scope^M, \bar{a}^M)$ be a model of the circumscription. We have to show that $\mathcal{M} \models \neg\tau(\bar{a})$. Assume that $\bar{a}^M \in \tau^M$. We will instantiate the second order variables in (15) in the following way: O^M for O', $\{\bar{a}^M\}$ for $Scope'$, $W(\beta', \delta')$ for τ', $\tau' \cap Scope'$ for X, β^M for β'.

For δ' we choose a binary relation on M in such a way, that the instantiated tuple $\bar{P}'$ makes $A(\bar{P}', Scope')$ true, but $\bar{a}^M \notin \tau'$. This can be achieved by changing the metric on the plane, but keeping collinearity. Alternatively, we can project plane M onto another plane that forms an appropriate angle with M.[2] In this way, we have the same triangles (in particular, $\bar{a}^M$ remains a triangle), but $\bar{a}^M$ does not have a right angle at vertex a.

It is easy to check that $A(\bar{P}', Scope')$ and $Def(\tau', Scope', X)$ are satisfied by these instantiations of the predicate variables. In addition to this, we obtain $X = \phi$, the empty set. Then, the antecedent of the implication in (16) becomes true. Therefore, the consequent of the implication is also true, contradiction.

□

5 Conclusions

5.1 Discussion: Why Is This Interesting?

- We have verified the suitability of scoped circumscription for a non-trivial, but not uninteresting domain, namely axiomatic geometry. In so doing, we have formally characterized the previously intuitive, but imprecise concept of a generic triangle (and by extension, of generic mathematical objects in general).

- A notion which has been implicit in this paper, but which deserves emphasis, is that *a generic is a binary relation between an individual and a set of properties.* One can only sensibly talk about objects generic with respect to some set of special properties. Thus, Theorem 2 states that the genericity of (a, b, c) with respect to the set of properties $\{(\lambda x, y, z)[x, y, z$ is a triangle right-angled at x$]\}$ is a circumscriptive consequence of elementary geometry. By extension, we

[2]This argument can be made precise by recalling that every model of this geometry is isomorphic to the cartesian product of a Euclidean field [21].

could also prove that the genericity of $\bar{a}$ with respect to the set of properties {*isosceles, right-angled*} is a circumscriptive consequence of elementary geometry. Since generics are relative to some choice of special properties to be minimized, we can easily obtain examples of conflicting defaults, as for example the triangles generic with respect to the properties {*scalene, obtuse*}.[3] Hence, it appears that even in the setting of axiomatic mathematics, many, if not all of the problems regarding multiple extensions and default priorities (Reiter and Criscuolo [14]) will arise. One advantage for studying generics in mathematics is that these are conceptually crisp domains (unlike royal elephants) so that "clashes of intuition" are less likely.

- One might object to our treatment of generics by arguing that it is sufficient, and certainly simpler, to write an axiom stating that the generic object is not defined by any of the singular predicates. There are two arguments against this objection. First, the circumscriptive approach provides a uniform and modular treatment of genericity. Modularity is achieved by virtue of the ease with which new singular predicates can be taken into account. More importantly, we cannot know in advance whether it can be consistently assumed that the generic object lies outside all the singular predicates. With the circumscriptive approach we are not led to the inconsistency that would be introduced by simply demanding that triangle $\bar{a}$ be neither obtuse nor scalene. Furthermore, we will frequently be seeking a generic object that satisfies some additional hypothesis, for example, a generic right-angled triangle in proving the Pythagorean Theorem. Under these circumstances, it is very likely that, without the circumscriptive approach, an inconsistent specification will be obtained.

- One of the main open problems in nonmonotonic reasoning concerns the design of general policies for applying nonmonotonic formalisms. Frequently, we do not know how to apply circumscription, that is, we do not know what to circumscribe and what form of circumscription to use. By starting with the most straightforward and naive applications of circumscription and showing why they are inadequate, and ending with

[3] A *scalene* triangle has an angle greater than a right angle. An *obtuse* triangle is one which is not scalene.

our succesful application of scoped circumscription, we hope to have shed some light on the problem of designing circumscriptive policies for different problems.

- Only the most determined probabilist would hold that a formal theory of generic mathematical objects must appeal to probabilities. Whatever the probability of such a generic, its real meaning is something like "not one of the special objects", where the "special" objects are axiomatically defined. It may well be that probability considerations had something to do with the first geometers focussing on isosceles triangles as something special, although surely their relative rarity among triangles is not all that makes them special. Nevertheless, once they have been labelled as special, which is to say once they have been axiomatically defined, the probability that a triangle be isosceles is forgotten, and only their specialness enters into the definition of a generic triangle. It seems to us that as goes the triangle, so goes Tweety.

- The most obvious application of a theory of generic mathematical objects is to automated deduction in mathematics where, in the style of Gelernter's Geometry Machine, a model is used in guiding the search for proofs (Gelernter [4], Reiter [12]). Frequently, mathematicians seek non-trivial models of some axiomatization, and the (semi-)automated generation of generic models would be an asset. More speculatively, we see applications of these ideas to computer graphics and problems like design layouts where the figures to be generated must be in "most general position".

5.2 Work in Progress and Future Research

- It would be interesting to explore generics in other mathematical domains, in a style parallel to that of this paper. For example, we are working on a circumscriptive characterization of the concept of *generic point of a curve* that appears in algebraic geometry [19, 20].[4] Since this concept has a precise definition in algebraic terms, it is a natural question whether it can be captured by means of a nonmonotonic logical specification. The definition of generic object given in algebraic

[4] We thank John McCarthy for suggesting this problem to us.

geometry has a pattern that occurs in other areas of mathematics. In [15, 16], Abraham Robinson generalizes the concepts of algebraic geometry, in particular, the concept of a generic point of a variety, to the context of model theory for deductive systems. In this work one finds the concepts of generic point, generic core, and generic structure. How are these notions related to nonmonotonic logics?

- Implementation techniques need investigation. In this connection, constraint logic programming appears to be a promising approach to the generation of generic geometric objects.

- How, if at all, is a formal theory of generic mathematical objects of interest to working logicians? Can nonmonotonic formalisms, originally conceived to deal with commonsense reasoning, contribute to modern logic?

Acknowledgments: Funding for this work was provided by the National Science and Engineering Research Council of Canada, the Institute for Robotics and Intelligent Systems, and the Catholic University of Chile (Grant DIUC 91-4E). The first author is also grateful to Javier Pinto and Rene Peralta for all their help and friendship provided during this research; and to the second author for motivation, enriching discussions and general support during his stay in Toronto.

References

[1] P. Cohen. *Set Theory and the Continuum Hypothesis*. Benjamin, 1966.

[2] J. Doyle. Circumscription and implicit definability. *Journal of Automated Reasoning*, 1:391–405, 1985.

[3] D.W. Etherington, S. Kraus, and D. Perlis. Nonmonotonicity and the scope of reasoning. *Artificial Intelligence*, 52:221–261, 1991.

[4] H. Gelernter. Realization of a geometry-theorem proving machine. In E.A. Feigenbaum and J. Feldman, editors, *Computers and Thought*, pages 134–152. McGraw-Hill Book Company, New York, 1963.

[5] P.C. Gilmore. An examination of the geometry theorem machine. *Artificial Intelligence*, 1:171–187, 1970.

[6] H.E. Kyburg Jr. *Probability and the Logic of Rational Belief.* Wesleyan University Press, 1961.

[7] V. Lifschitz. Computing circumscription. In *Proceedings of the Ninth International Joint Conference on Artificial Intelligence*, pages 121–127. Morgan Kaufmann Publishers, Inc., 1985.

[8] J. McCarthy. Circumscription - a form of non-monotonic reasoning. *Artificial Intelligence*, 13:27–39, 1980.

[9] J. McCarthy. Applications of circumscription to formalizing commonsense knowledge. *Artificial Intelligence*, 28:89–116, 1986.

[10] D. Poole. Explanation and prediction: an architecture for default and abductive reasoning. *Computational Intelligence*, 5:97–110, 1989.

[11] A. Quaife. Automated development of Tarski's geometry. *Journal of Automated Reasoning*, 5:97–118, 1989.

[12] R. Reiter. A semantically guided deductive system for automatic theorem-proving. *IEEE Transactions on Computers*, C-25:328–334, 1976.

[13] R. Reiter. Nonmonotonic reasoning. *Annual Reviews in Computer Science*, 2:147–186, 1987.

[14] R. Reiter and G. Criscuolo. Some representational issues in default reasoning. *International Journal of Computers and Mathematics with Applications*, 9:15–27, 1983.

[15] A. Robinson. *Théorie Métamathématique des Ideaux.* Gauthier-Villars, Paris, 1955.

[16] A. Robinson. *Introduction to Model Theory and to the Metamathematics of Algebra.* North-Holland, 1974.

[17] H.L. Royden. Remarks on primitive notions for elementary Euclidean and non-Euclidean plane geometry. In L. Henkin, P. Suppes, and A. Tarski, editors, *The Axiomatic Method.* North-Holland, 1959.

[18] D. Scott. A symmetric primitive notion for Euclidean geometry. *Indagationes Mathematicae*, 18:457–461, 1956.

[19] A. Seidenberg. *Elements of the Theory of Algebraic Curves.* Addison-Wesley, 1968.

[20] J.G. Semple and G.T. Kneebone. *Algebraic Curves*. Oxford University Press, 1959.

[21] A. Tarski. What is elementary geometry? In L. Henkin, P. Suppes, and A. Tarski, editors, *The Axiomatic Method*. North-Holland, 1959.