Proximity Constraints and Representable Trees*

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Abstract

This paper examines an infinite family of proximity drawings of graphs called *open and closed β-drawings*, first defined by Kirkpatrick and Radke [15, 21] in the context of computational morphology. Such proximity drawings include as special cases the well-known Gabriel, relative neighborhood and strip drawings. Complete characterizations of those trees that admit open β-drawings for $0 \le \beta \le \frac{1}{1-\cos(\frac{2\pi}{5})}$ and $\frac{1}{\cos(\frac{2\pi}{5})} < \beta < \infty$ or closed β-drawings for $0 \le \beta < \frac{1}{1-\cos(\frac{2\pi}{5})}$ and $\frac{1}{cos(\frac{2\pi}{5})} \le \beta \le \infty$ are given, as well as partial characterizations for other values of β. For the intervals of β in which complete characterizations are given, it can be determined in linear time whether a tree admits an open or closed β-drawing, and, if so, such a drawing can be computed in linear time in the real RAM model. Finally, a complete characterization of all graphs which admit closed strip drawings is given.

1 Introduction and Overview

A drawing of a graph G maps the vertices of G to distinct points in the plane and each edge (u, v) of G to a simple curve between the points associated with u and v. Graph drawing algorithms and tools usually adopt given graphic standards. A widely used graphic standard represents all the edges as straight-line segments. Drawings within this standard are called *straight-line drawings*. A limited list of work on straight-line drawings includes [11, 12, 13, 14, 23].

Increasing attention has been given recently to *proximity drawings* [2, 16, 10, 19, 9]; for a survey on proximity drawings the reader is referred to [5]. Loosely speaking, a proximity drawing is one in which adjacent vertices are drawn relatively close together and non-adjacent vertices are drawn relatively far apart. Many types of proximity drawings measure relative closeness by using a *proximity region*: Given two points u and v in the plane, a proximity region for u and v is a subset of the plane associated with u and v. A proximity drawing of G is a straight-line drawing such that: (i) for each edge (u, v) of G, the proximity region of the points representing u and v is empty (does not contain any other vertex); and (ii)

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for each pair of nonadjacent vertices u, v of G, the proximity region of the points representing u and v is not empty.

Several types of proximity regions have been investigated, each one chosen for particular application purposes. Examples include

1. the *relative neighborhood region*: the intersection of the two open disks centered at u and at v and with distance $d(u, v)$ as radius;
2. the *Gabriel region*: the closed disk having u and v as antipodal points; and
3. the *closed strip region*: the infinite closed strip having u and v on the boundary and width $d(u, v)$.

For example, in Figure 1.a we show the proximity drawing of a tree T where the proximity regions are relative neighborhood regions. Observe that T contains edge (x, z) and the proximity region of the pair x, z is empty; conversely edge (w, v) is not in T and the proximity region of w, v contains x ($\{x, w, v\}$ were chosen to make angle $\angle wxv$ the smallest of the five angles). Tree T has no proximity drawing such that the proximity regions are Gabriel regions. Figure 1.b shows a proximity drawing of another tree T', using Gabriel regions. Figure 1.c shows that the same drawing is also a closed strip proximity drawing of T'.

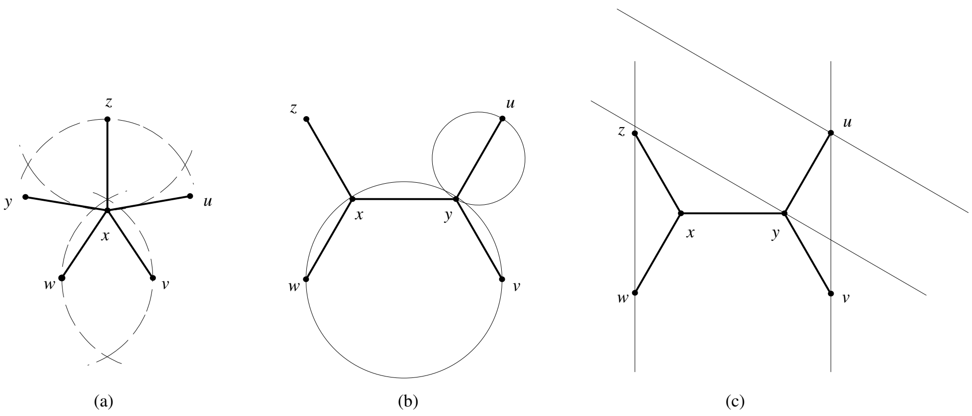

Figure 1: Three proximity drawings.

In this paper we study the *proximity-drawability testing problem*: the problem of deciding whether a graph has a proximity drawing with a given type of proximity region. In particular we study the proximity-drawability of trees. We consider an infinite parametrized family of proximity regions, first introduced in [15, 21], that includes several of the most well-known proximity regions from the literature.

We consider two types of proximity region:

Definition 1 Given a pair x, y of points in the plane, the *open β-region of x and y*, and the *closed β-region of x and y*, denoted by $R(x, y, \beta)$ and $R[x, y, \beta]$ respectively, are defined as follows:

1. For $0 < \beta < 1$, $R(x, y, \beta)$ is the intersection of the two open disks of radius $d(x, y)/(2\beta)$ passing through both x and y. $R[x, y, \beta]$ is the intersection of the two corresponding closed disks.

2. For $1 \leq \beta < \infty$, $R(x, y, \beta)$ is the intersection of the two open disks of radius $\beta d(x, y)/2$ and centered at the points $(1 - \beta/2)x + (\beta/2)y$ and $(\beta/2)x + (1 - \beta/2)y$. $R[x, y, \beta]$ is the intersection of the two corresponding closed disks.

3. $R(x, y, \infty)$ is the open infinite strip perpendicular to the line segment $\overline{xy}$ and $R[x, y, \infty]$ is the closed infinite strip perpendicular to the line segment $\overline{xy}$.

4. Finally, $R(x, y, 0)$ is the empty set and $R[x, y, 0]$ is the line segment connecting x and y.

Figure 2 illustrates some $[\beta]$-regions of a pair of points $\{x, y\}$ for different values of β. In Figure 1, $R(x, z, 2)$, $R[u, y, 1]$, and $R[w, v, \infty]$ are examples of the relative neighborhood region, Gabriel region, and closed strip region, respectively.

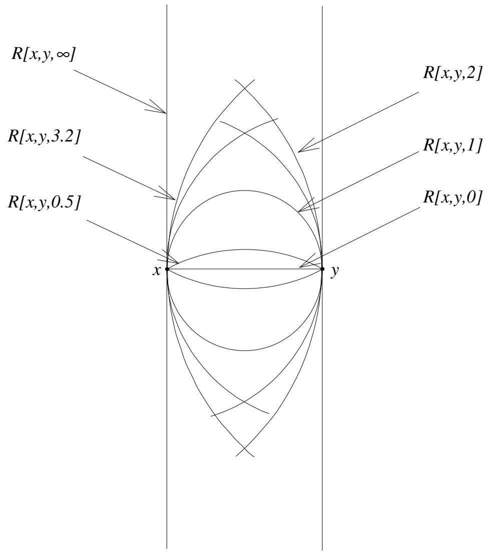

Figure 2: A set of proximity regions $R[x, y, \beta]$

1.1 Applications

The problem of testing whether a tree has a proximity drawing and, if so, of constructing such a drawing has applications in the area of graph drawing. The design of algorithms for straight-line drawings of trees is a field of growing interest given the ubiquity of trees as models. For a small sample of papers that show algorithms for straight-line drawings of trees see [8, 7, 2]. Proximity drawings of trees have several interesting characteristics for visualization:

1. Neighbors of a given vertex cluster around that vertex;

2. The angles between consecutive edges are "large" (each angle is at least $\pi/3$); and

3. Proximity drawings of trees, as we will see later, are related to minimum spanning trees, another well studied class of tree-drawings [19, 9].

Note that the problem of constructing drawings with large angles (high-resolution drawings) has been studied in [17, 6]. For an up to date overview on graph drawing problems, applications, and algorithms, the reader is referred to [4].

Finally, proximity drawing problems may be viewed as visibility problems: two points are mutually visible if a certain region between them contains no other point. From this point of view, the results in this paper deal with the problem of determining whether a tree can be realized as the visibility tree of a set of points.

1.2 Results

Let $\mathcal{T}(\beta)$ ($\mathcal{T}[\beta]$) be the class of trees that have a proximity drawing where the proximity region is the open (closed) β-region. We denote with $\mathcal{T}_k$ the set of all finite trees of maximum vertex degree at most k. Class $\overline{\mathcal{T}}$ is defined in Section 5 and class $\overline{\overline{\mathcal{T}}}$ are the so-called "forbidden" graphs defined in [2]. The results presented in this paper are listed below. Table 1 summarizes the characterization results and compares them with previous results, showing how the set of drawable trees changes as β changes. Columns of the table labelled "new" describe results of this paper; Columns labelled "previous" describe known results. A citation indicates that the result either first appeared in—or is a simple consequence of results appearing in—the cited papers.

- We give a complete characterization of $\mathcal{T}(\beta)$ for all β values such that $0 \leq \beta \leq \frac{1}{1-cos(\frac{2\pi}{5})} \simeq 1.45$ or such that $3.23 \simeq \frac{1}{\cos(\frac{2\pi}{5})} < \beta < \infty$. Also, we give a complete characterization of $\mathcal{T}[\beta]$ for all β values such that $0 \leq \beta < \frac{1}{1-cos(\frac{2\pi}{5})}$ or such that $\frac{1}{\cos(\frac{2\pi}{5})} \leq \beta \leq \infty$. For all β values not in the above intervals, we give a partial characterization: we show that all trees in $\mathcal{T}_4$ and only trees in $\mathcal{T}_5$ belong to $\mathcal{T}(\beta)$ and $\mathcal{T}[\beta]$.
- For any β in the intervals mentioned above, we can in linear time decide membership in $\mathcal{T}(\beta)$ or $\mathcal{T}[\beta]$.
- We describe linear time algorithms (in the real-RAM model), which, given any β in the intervals mentioned above, and any tree $T \in \mathcal{T}(\beta)$ (or $\mathcal{T}[\beta]$), construct a proximity drawing of T with proximity region the open (or closed) β-region. Furthermore, we can produce in linear time such a proximity drawing for any tree in $\mathcal{T}_4$ and any value of β such that $1.45 < \beta < 3.23$.
- We discuss the relationships between the proximity drawings presented in this paper, Delaunay triangulations, and minimum spanning trees and exploit these relationships in our proof techniques.
- Furthermore, we show that the class of graphs that can be drawn with proximity region $R[x, y, \infty]$ consists of all binary forests. To date, this is the first complete characterization of a class of proximity drawable graphs.

The paper is organized as follows. Section 2 contains basic terminology. Geometric properties of proximity drawings adopting (β)- and [β]-proximity regions, as well as their combinatorial relationships to minimum spanning trees and Delaunay triangulations are discussed in Section 3. Section 4 presents proximity drawing algorithms for different classes of trees. All the described algorithms produce drawings with the property that for any two non-adjacent vertices u, v, the β-region of u and v contains some vertex

	β	$\mathcal{T}(\beta)$ previous	$\mathcal{T}[\beta]$ previous	$\mathcal{T}(\beta)$ new	$\mathcal{T}[\beta]$ new
1	$\beta = 0$	—	—	$\mathcal{T}(\beta) = \{K_1, K_2\}$	$\mathcal{T}[\beta] = \mathcal{T}_2$
2	$0 < \beta < \frac{\sqrt{3}}{2}$	—	—	$\mathcal{T}(\beta) = \mathcal{T}_2$	$\mathcal{T}[\beta] = \mathcal{T}_2$
3	$\beta = \frac{\sqrt{3}}{2}$	—	—	$\mathcal{T}(\beta) = \mathcal{T}_2$	$\mathcal{T}[\beta] = \mathcal{T}_3 - \overline{\mathcal{T}}$
4	$\frac{\sqrt{3}}{2} < \beta < 1$	—	—	$\mathcal{T}(\beta) = \mathcal{T}_3$	$\mathcal{T}[\beta] = \mathcal{T}_3$
5	$\beta = 1$	$\mathcal{T}(\beta) = \mathcal{T}_3$ [2]	$\mathcal{T}[\beta] = \mathcal{T}_4 - \overline{\overline{\mathcal{T}}}$ [2]	—	—
6	$1 < \beta < \frac{1}{1-\cos(\frac{2\pi}{5})}$	$\mathcal{T}_3 \subseteq \mathcal{T}(\beta)$ [3, 15]	$\mathcal{T}_3 \subseteq \mathcal{T}[\beta]$ [24, 15]	$\mathcal{T}(\beta) = \mathcal{T}_4$	$\mathcal{T}[\beta] = \mathcal{T}_4$
7	$\beta = \frac{1}{1-\cos(\frac{2\pi}{5})}$	$\mathcal{T}_3 \subseteq \mathcal{T}(\beta)$ [3, 15]	$\mathcal{T}_3 \subseteq \mathcal{T}[\beta]$ [24, 15]	$\mathcal{T}(\beta) = \mathcal{T}_4$	$\mathcal{T}_4 \subset T[\beta] \subset \mathcal{T}_5$
8	$\frac{1}{1-\cos(\frac{2\pi}{5})} < \beta < 2$	$\mathcal{T}_3 \subseteq \mathcal{T}(\beta)$ [3, 15]	$\mathcal{T}_3 \subseteq \mathcal{T}[\beta]$ [24, 15]	$\mathcal{T}_4 \subset \mathcal{T}(\beta) \subseteq \mathcal{T}_5$	$\mathcal{T}_4 \subset T[\beta] \subseteq \mathcal{T}_5$
9	$\beta = 2$	$\mathcal{T}(\beta) = \mathcal{T}_5$ [2]	$\mathcal{T}[\beta] = \mathcal{T}_5$ [2]	—	—
10	$2 < \beta < \frac{1}{\cos(\frac{2\pi}{5})}$	—	—	$\mathcal{T}_4 \subset \mathcal{T}(\beta) \subseteq \mathcal{T}_5$	$\mathcal{T}_4 \subset T[\beta] \subseteq \mathcal{T}_5$
11	$\beta = \frac{1}{\cos(\frac{2\pi}{5})}$	—	—	$\mathcal{T}_4 \subset \mathcal{T}(\beta) \subset \mathcal{T}_5$	$\mathcal{T}[\beta] = \mathcal{T}_4$
12	$\frac{1}{\cos(\frac{2\pi}{5})} < \beta < \infty$	—	—	$\mathcal{T}(\beta) = \mathcal{T}_4$	$\mathcal{T}[\beta] = \mathcal{T}_4$
13	$\beta = \infty$	—	—	$\mathcal{T}_3 \subset \mathcal{T}(\beta) \subset \mathcal{T}_4$	$\mathcal{T}[\beta] = \mathcal{T}_3$

Table 1: Summarizing the characterization results

along the path from u to v. A further contribution of Section 4 is to show that for $1.45 < \beta < 2$ there exist trees in $\mathcal{T}_5$ that do not admit proximity drawings having such property. Characterizations of representable trees for different values of β are given in Section 5. Graphs that admit a closed strip proximity drawing are characterized in Section 6. Finally, Section 7 discusses conclusions and open problems.

2 Preliminaries

We assume familiarity with the basic terminology of graph theory and computational geometry (see also [1], [20]). Let T be a tree, v a vertex of T, and x a neighbor of v; $T_x(v)$ denotes the connected component of $T - \{v\}$ which contains x. A *rooted tree* (T, r) is a tree T along with a distinguished vertex r (called the *root* of T). Vertices adjacent to r are called *children* of r and r is the *parent* of those vertices. Each child x of r is in turn the root of $T_x(r)$.

Any closed sector of a disk with center p and radius ρ of angle $\phi < 2\pi$ is denoted by $S[p, \phi, d, \rho]$, where d is a vector with origin p, which bisects the angle ϕ of the sector. A sector having angle ϕ is called an *ϕ-sector.*

Given a rooted tree (T, r), a β-drawing of (T, r) is a *S_ϕ-drawing* if all the vertices of the drawing are contained in an ϕ-sector having r drawn at the apex. Note that the radius of the sector is irrelevant; if (T, r) has some S_ϕ-drawing, it can be drawn in any ϕ-sector of any positive radius.

Let S, X and Y be any three sets of points in the plane. We say that S *separates* X and Y if, for each $x \in X$ and each $y \in Y$, where $x \neq y$, there is a point in S which lies in the proximity region of x and y. Thus if S' is a set of points that contains S, $x \in X \cap S'$ and $y \in Y \cap S'$, then a proximity drawing whose vertices are the points of S' will not contain the edge between x and y. Some of our algorithms will proceed by representing disjoint subtrees of a tree inside sectors that are separated from each other by some set of points.

An *open β-drawing* (or *(β)-drawing*) of a graph G is a proximity drawing of G such that for each pair of points x, y the proximity region is $R(x, y, \beta)$. Analogously, a *closed β-drawing* (or *$[\beta]$-drawing*) of G is a proximity drawing of G such that for each pair of points x, y the proximity region is $R[x, y, \beta]$.

A graph is *(β)-drawable* if it has a (β)-drawing. A class of graphs is (β)-drawable if each graph in the class is (β)-drawable. A class of graphs is not (β)-drawable if it contains at least one graph that is not (β)-drawable. Similar terminology and notation is used for closed β-drawings. When it is clear from the context or when it is not necessary to distinguish between open and closed proximity regions, we will simplify the notation by talking about *β-drawings* and *β-drawable* graphs and classes. For brevity, we will sometimes use the term *β-graph* instead of β-drawable graph.

Given a set P of points in the plane, we denote by $G(P, \beta)$ the graph whose vertices correspond to the points of P and such that there is an edge (x, y) between two vertices corresponding to points x and y iff $R(x, y, \beta) \cap P = \emptyset$. It is easy to see that $G(P, \beta)$ has a (β)-drawing that is obtained by connecting with straight-line segments the points of P that correspond to adjacent vertices of $G(P, \beta)$. Hence, $G(P, \beta)$ is a (β)-graph. For convenience we denote, where this does not cause ambiguity, by $G(P, \beta)$ both the graph and its (β)-drawing and by P both the set of vertices and the points representing them in the drawing. Analogously, we denote by $G[P, \beta]$ the graph whose vertices correspond to the points of P and such that there is an edge between two vertices x and y iff $R[x, y, \beta] \cap (P - \{x, y\}) = \emptyset$. Clearly, $G[P, \beta]$ has a $[\beta]$-drawing that is obtained by connecting with straight-line segments the vertices of P that correspond to adjacent vertices. Hence, $G[P, \beta]$ is a $[\beta]$-graph. Also in this case we will sometimes denote by $G[P, \beta]$ both the graph and its drawing.

An induced subgraph of a graph G which is obtained by repeated removal of leaves (i.e. vertices of degree one) is called a *pruning* of G. Let G be a graph which admits a β-drawing Γ and let G' be a pruning of G obtained by removing a set of vertices V'. Let Γ' be obtained from Γ by removing the points corresponding to the set V'. If for all prunings G' of G, Γ' is a β-drawing of G', then Γ is a *stable* β-drawing of G.

Property 1 *In a stable β-drawing of a tree, for any pair of non-adjacent vertices x and y there is a vertex v on the (unique) path between x and y such that v is contained in the proximity region of x and y.*

Figure 3.a and 3.b show two different [2]-drawings of the same tree. The first drawing is [2]-stable, the second drawing is not [2]-stable.

To analyze β-drawings we will frequently use two angles $\alpha(\beta)$ and $\gamma(\beta)$, defined as follows.

1. $\alpha(\beta) = \inf\{\angle xzy \parallel z \in R[x, y, \beta]\}$. See Figure 4.a.
2. $\gamma(\beta)$ is only defined for $\beta \geq 2$, and $\gamma(2) = \frac{\pi}{3}$. For $\beta > 2$, let $z \neq y$ be a point on the boundary of $R[x, y, \beta]$ such that $d(x, y) = d(x, z)$. Then $\gamma(\beta) = \angle zxy$. See Figure 4.b.

Note that $\alpha(0) = \pi$, and that the value of $\alpha(\beta)$ decreases as β increases; for example, $\alpha(1) = \pi/2$, $\alpha(2) = \pi/3$, and $\alpha(\infty) = 0$. Conversely, $\gamma(\beta)$ increases from $\pi/3$ to $\pi/2$ as β increases from 2 to ∞.

When the value of β is understood, we will often write α and γ instead of $\alpha(\beta)$ and $\gamma(\beta)$. The following property shows how β is related to α and γ and can be proved by means of elementary geometric arguments.

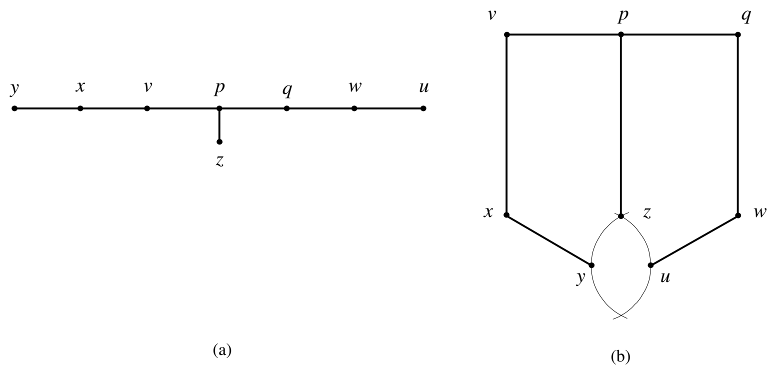

Figure 3: (a) A stable [2]-drawing and (b) a non-stable [2]-drawing of the same tree.

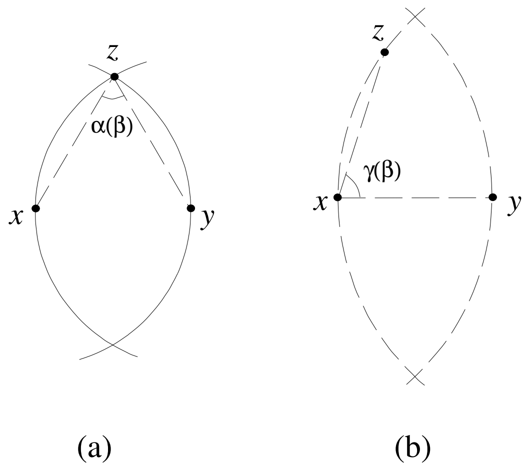

Figure 4: angles $\alpha(\beta)$ and $\gamma(\beta)$.

Property 2

1. *$\beta = \sin\alpha$ for $0 \le \beta < 1$.*
2. *$\beta = \frac{1}{1-\cos\alpha}$ for $1 \le \beta \le \infty$.*
3. *$\beta = \frac{1}{\cos\gamma}$ for $2 \le \beta \le \infty$.*

A *Delaunay triangulation* of P, denoted by $DT(P)$, is a planar graph whose vertices correspond to the points of P and whose edges are defined as follows. Construct a triangulation of P such that each interior triangle has the property that the open disk circumscribing the triangle contains no other point of P. The edges of $DT(P)$ are the edges of the triangles. A set P may admit more than one Delaunay triangulation, but only if P contains four or more co-circular points. Obviously, the described triangulation of P is a planar straight-line drawing of $DT(P)$. We will sometimes denote by $DT(P)$ both the graph and the drawing.

A *minimum spanning tree* of P, denoted by $MST(P)$, is a spanning tree of P of minimum total edge length. In general, a set P may have many minimum spanning trees (for example, if P consists of the vertices of a regular polygon). Again we will sometimes denote by $MST(P)$ both the graph and the drawing.

3 Points, Graphs, and Drawings

Here we study some basic properties of β-graphs and β-drawings, and their relation to minimum spanning trees and Delaunay triangulations. In the following P denotes a finite set of points in the plane.

3.1 Properties of β-drawings

For a given $\beta \geq 0$, the open β-region is strictly contained in the closed one. Therefore, every edge in $G[P, \beta]$ is also an edge of $G(P, \beta)$. Also, an open (and closed) β-region is strictly contained in every other β'-region for which $\beta' > \beta$. Thus, every edge in $G(P, \beta')$ is also an edge in $G[P, \beta]$. We summarize this in the following property.

Property 3 *If* $0 \leq \beta_1 < \beta_2 \leq \infty$ *then* $G[P, \beta_2] \subseteq G(P, \beta_2) \subseteq G[P, \beta_1] \subseteq G(P, \beta_1)$.

Figure 5 shows a set of points P and some different graphs $G(P, \beta)$ as β ranges from 0 to ∞.

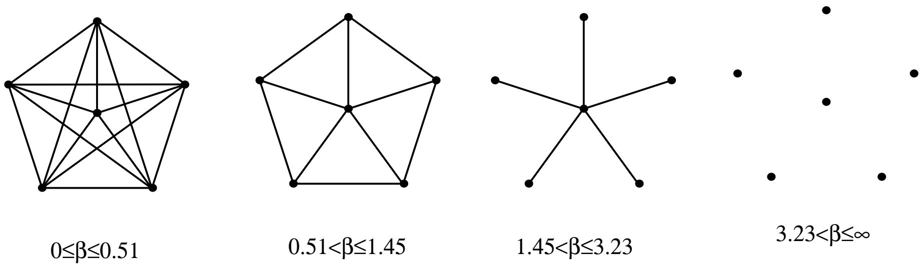

Figure 5: Different $G(P, \beta)$ as β varies.

Property 4 *For* $\beta > 1$, $G(P, \beta)$ *and* $G[P, \beta]$ *are planar; also,* $G[P, 1]$ *is planar, but* $G(P, 1)$ *is not necessarily planar. For* $\beta < 1$, $G(P, \beta)$ *and* $G[P, \beta]$ *are not necessarily planar.*

Proof: The first part easily follows from the planarity of $G[P, 1]$ that is proved in [18] and from Property 3. The second part is proved as follows. Suppose P consists of exactly six points and that they are placed at the vertices of a regular hexagon (see Figure 6.a). Clearly, $G(P, 1)$ is a $K_{3,3}$; from Property 3 the statement follows. □

(a) (b)

Figure 6: (a)$G(P,1)$ is non-planar; (b) $G[P,2]$ is disconnected.

Property 5 *For $\beta < 2$, $G(P,\beta)$ and $G[P,\beta]$ are connected. Also, $G(P,2)$ is connected, but $G[P,2]$ is not necessarily connected. For $\beta > 2$, $G(P,\beta)$ and $G[P,\beta]$ are not necessarily connected.*

Proof: The first part easily follows from the connectivity of $G(P,2)$ that is proved in [22] and from Property 3. The second part is proved as follows. Suppose P consists of six points placed at the vertices of a regular hexagon plus a seventh point at the center of the hexagon (see Figure 6.b). Clearly, $G[P,2]$ is disconnected; from Property 3 the statement follows. $\square$

The following lemma deals with the β-drawability of trees. It gives a tool to extend the result on the β-drawability of a tree for a given value of β to infinitely many other values of β.

Lemma 1 *Let β be such that $0 \leq \beta < 2$. If $G(P,\beta)$ is a tree, then $G(P,\beta') = G[P,\beta'] = G(P,\beta)$ for all $\beta \leq \beta' < 2$. Also, $G(P,2) = G(P,\beta')$.*

Proof: We prove that $G(P,\beta') = G(P,\beta)$; the proof that $G[P,\beta'] = G(P,\beta)$ is analogous. Property 3 implies that $G(P,\beta') \subseteq G(P,\beta)$. Since $G(P,\beta)$ is a tree and $G(P,\beta')$ is connected by Property 5, $G(P,\beta')$ is also a tree, and so is equal to $G(P,\beta)$. $\square$

3.2 β-drawings and Delaunay Triangulations

Property 6 *For $\beta > 1$, $G(P,\beta)$ and $G[P,\beta]$ are subgraphs of $DT(P)$. Also, $G[P,1]$ is a subgraph of $DT(P)$.*

Proof: The fact that $G[P,1]$ is a subgraph of $DT(P)$ is proved in [18]. The rest follows from Property 3. $\square$

From Property 4 and from the planarity of $DT(P)$ it follows that for $\beta < 1$ the above property, in general, does not hold.

Lemma 2 and Theorem 1 generalize to infinitely many values of β similar results that have been given in [2] for $\beta = 1$. Since the proofs are based on an analogous reasoning as in [2], we only briefly sketch them.

Lemma 2 *Let $0 \leq \beta \leq 1$ and let (u,v) be an edge of $DT(P)$. Edge $(u,v) \in G[P,\beta]$ if and only if for each triangle $\triangle uxv$ in $DT(P)$, $\angle uxv < \alpha(\beta)$. The same result holds for $G(P,\beta)$ but with $\angle uxv \leq \alpha(\beta)$.*

Proof: We consider only the case of $G[P,\beta]$ when (u,v) is shared by two triangles $\triangle ucv$ and $\triangle udv$ of $DT(P)$. Assume that (u,v) is in $G[P,\beta]$. Since $R[u,v,\beta]$ is empty, for all $x \in P$, we have $\angle uxv < \alpha$. Conversely, if $\angle ucv$ and $\angle udv$ are both less than α, since $\beta < 1$ we have that $R[u,v,\beta]$ is contained in the union of the two disks circumscribing $\triangle ucv$ and $\triangle udv$. Thus $R[u,v,\beta]$ is empty and so (u,v) is in $G[P,\beta]$. □

Theorem 1 *Let $0 \leq \beta \leq 1$. If $G[P,\beta]$ is a tree, then for each cycle C of $DT(P)$ there exists an edge $(u,v) \in C$ not in $G[P,\beta]$ such that for some point $p \in P$, $\triangle upv$ is a face of $DT(P)$, $\triangle upv$ lies inside C, and $\angle upv \geq \alpha(\beta)$. The same result holds for $G(P,\beta)$ but with $\angle upv > \alpha(\beta)$.*

Proof: We show the proof for $[\beta]$-regions. The reasoning for open regions is analogous. Consider any two cycles C and C' in $DT(P)$. We will say that C is *contained* in C' if the region bounded by C' contains the region bounded by C. This gives a partial order on the cycles of $DT(P)$. Since $G[P,\beta]$ is a tree, any cycle C must contain an edge that is not in $G[P,\beta]$. Consider then the set of all cycles which do not satisfy the conclusion of the theorem. If this set is empty, then we are done; otherwise this set contains a cycle C which is maximal with respect to containment, i.e. there does not exist another cycle in the set that contains C. Consider an edge (u,v) on C which is not in $G[P,\beta]$. By Lemma 2 it must lie on a triangle $\triangle upv$ of $DT(P)$ such that $\angle upv \geq \alpha(\beta)$. Since (u,v) does not satisfy the conclusion of the theorem, $\triangle upv$ must lie outside C. Let C' be the smallest (in terms of containment) cycle of $DT(P)$ which contains both C and $\triangle upv$. Then the edges of C' consist of edges of C and at least one of (u,p) and (v,p). Thus C' violates the conclusion of the theorem, since the edges (u,p) and (p,v) do not satisfy the conclusion of the theorem ($\triangle upv$ is a triangle of $DT(P)$ that lies inside C', where the angle with apex u and the one with apex v are both less than $\alpha(\beta)$ since, by Property 2, $\alpha(\beta) \geq \pi/2$ when $0 \leq \beta \leq 1$). This, however, contradicts the maximality of C. □

3.3 β-drawings and Minimum Spanning Trees

We now discuss the relation between minimum spanning trees and β-drawings. The implications of this relationship are useful for the analysis of β-drawability.

Theorem 2

1. *$MST(P) \subseteq G(P,\beta)$ and $MST(P) \subseteq G[P,\beta]$ for $0 \leq \beta < 2$. Also $MST(P) \subseteq G(P,2)$.*
2. *There exists a set P such that for all β such that $2 < \beta < \infty$, $G[P,2] = G[P,\beta] = G(P,\beta) = G(P,\infty)$ is a tree but is not $MST(P)$.*

Proof: The first part of Statement 1 is a consequence of Property 3; the second part is proved in [22]. To prove Statement 2, consider the drawing in Figure 7. Observe that this is a β-drawing for any $2 < \beta < \infty$ and that it is both a $[2]$-drawing and a (∞)-drawing. Vertices y, z, u form an equilateral triangle. Vertices x, z, w lie on a (horizontal) line parallel to the line determined by vertices y, u so that the triangles x, y, z and u, w, z are right triangles. Vertices v, s, t, q, p are far enough vertically above vertices x, y, z, u, w,

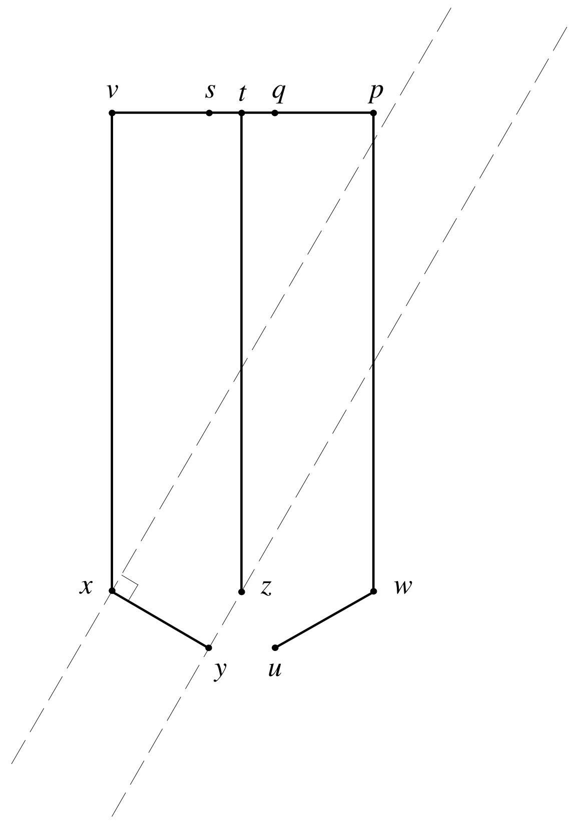

Figure 7: The drawing Γ in the proof of Theorem 2.

respectively, so that none of them are in either $R(x, y, \infty)$ or $R(u, w, \infty)$. Clearly $d(y, u) < d(x, v)$, so the drawing is not a minimum spanning tree. $\square$

The relationship between $G[P, \infty]$ and $MST(P)$ is discussed in Section 6. Furthermore, the following property is proved in [15].

Lemma 3 *[15] For each edge $e \in G(P, \infty)$, there exists a minimum spanning tree of P containing e.*

The relationship between β-graphs and minimum spanning trees can be exploited to give a lower bound for the minimum angle between consecutive edges in a β-drawing of a tree.

Lemma 4 *Assume $G(P, \beta)$ is a tree. For any value of β, the angle between any two consecutive edges of $G(P, \beta)$ is greater than $\alpha(\beta)$. If $2 < \beta \leq \infty$, this angle is at least $\gamma(\beta)$.*

Proof: We start by proving the latter part of the lemma. For $2 < \beta$, consider any two consecutive edges (u, z) and (z, v) of $G(P, \beta)$. Suppose that (u, z) is at least as long as (z, v). If $\angle uzv < \gamma(\beta)$, then v is in $R(u, z, \beta)$, contradicting the assumption that, (u, z) is an edge of $G(P, \beta)$. Since $\gamma(\beta) > \alpha(\beta)$ for all $\beta > 2$, we need only consider values of β such that $0 \leq \beta \leq 2$ in order to complete the proof.

We proceed by contradiction. Suppose v, x, y are vertices in P such that $(v,x), (v,y)$ are in $G(P,\beta)$ and $\angle xvy \leq \alpha(\beta)$. Let T_x and T_y be the subtrees, containing x and y respectively, obtained by removing v from $G(P,\beta)$.

Let $x' \in T_x$ and $y' \in T_y$ be two vertices such that

$$d(x', y') \leq d(\overline{x}, \overline{y}), \forall \overline{x} \in T_x \text{ and } \forall \overline{y} \in T_y.$$

$R(x', y', \beta)$ does not contain any points of T_x or T_y since for $\beta \leq 2$, if $z \in R(x'y', \beta)$, then either $d(z, x') < d(x', y')$ or $d(z, y') < d(x'y')$ contradicting the minimality of $d(x', y')$. It must, however, contain some point z in $G(P,\beta) - T_x - T_y$ since there is not an edge between x' and y' in $G(P,\beta)$. $G(P,\beta)$ is a tree, thus there must be a unique path from v to z, denoted by $P(v,z)$. This path contains no vertices of T_x or T_y, since z is in a connected subgraph of $G(P,\beta) - v$ different from both T_x and T_y. We can therefore conclude the following:

1. Since $G(P,\beta)$ is a minimum spanning tree, by Theorem 2 and since $P(v,z) \cup P(v,x') \cup x'z$ forms a cycle where every edge is contained in $G(P,\beta)$ except for (x', z), we have that $d(x', z) > d(x, v)$.
2. $d(y', z) > d(y, v)$ by the same argument as above.
3. $d(x', y') \leq d(x, y)$ by the relation between x' and y'.
4. $\angle x'zy' > \alpha(\beta)$ since z is contained in $R(x', y', \beta)$.

The above four inequalities, along with the fact that $\beta \leq 2$, allow us to conclude that point v lies inside $R(x, y, \beta)$, which implies $\angle xvy > \alpha(\beta)$, a contradiction. $\square$

With similar reasoning we can prove:

Lemma 5 *Assume $G[P,\beta]$ is a tree. For all values of β, the angle between any two consecutive edges of $G[P,\beta]$ is at least $\alpha(\beta)$. If $2 \leq \beta \leq \infty$, the angle is greater than $\gamma(\beta)$.*

Lemmas 4 and 5 imply the following theorem on the maximum vertex degree of β-drawings of trees.

Theorem 3

1. *If $G(P,\beta)$ is a tree then the maximum vertex degree of $G(P,\beta)$ is at most*
   (a) $\lceil \frac{2\pi}{\alpha(\beta)} \rceil - 1$ *if* $0 \leq \beta \leq 2$,
   (b) $\lfloor \frac{2\pi}{\gamma(\beta)} \rfloor$ *if* $2 < \beta \leq \infty$.
2. *If $G[P,\beta]$ is a tree then the maximum vertex degree of $G[P,\beta]$ is at most*
   (a) $\lfloor \frac{2\pi}{\alpha(\beta)} \rfloor$ *if* $0 \leq \beta < 2$,
   (b) $\lceil \frac{2\pi}{\gamma(\beta)} \rceil - 1$ *if* $2 \leq \beta \leq \infty$.

From Property 2 it follows that $\alpha(\beta) \geq \pi/3$ for $0 \leq \beta \leq 2$, and that $\gamma(\beta) \geq \pi/3$ for $2 \leq \beta \leq \infty$. Hence, the above theorem has the following consequence.

Corollary 1 *For all values of β, β-trees do not have vertices of degree greater than five.*

This corollary allows us to restrict our attention to the β-drawability properties of classes of trees $\mathcal{T}_k$ with $k < 6$.

4 Classes of β-trees

Given any positive value of β, one can always construct a β-drawing of any path (i.e. any element of class $\mathcal{T}_2$) by representing the vertices of the path as points ordered on a line. Clearly, this construction is also a [0]-drawing of the path. It is worth noting that no tree on three or more vertices is (0)-drawable, since $G(P, 0)$ is a clique for any set P of points. This discussion is summarized in the following theorem.

Theorem 4 *Class $\mathcal{T}_2$ is (β)-drawable for all values of $\beta > 0$ and is [β]-drawable for all values of β. Class $\mathcal{T}_2$ is not* (0)*-drawable.*

In the rest of the section we show intervals of representability and drawing algorithms for the classes T_3 and T_4; we also present negative results on the β-drawability of class T_5.

4.1 The Class $\mathcal{T}_3$

Lemma 6 *Let T be any tree in $\mathcal{T}_3$ and let β' be such that $\frac{\sqrt{3}}{2} < \beta' \leq \infty$. There exists a set P of points in the plane such that for each β, $\beta' \leq \beta \leq \infty$, $G(P, \beta)$ is a stable (β)-drawing of T and $G[P, \beta]$ is a stable [β]-drawing of T.*

Proof: The proof is by induction on the number of vertices in T. Let β' be such that $\frac{\sqrt{3}}{2} < \beta' \leq \infty$. It is clear that both the tree consisting of a single vertex and the tree consisting of a vertex of degree three and three leaves can be drawn satisfying the lemma. Assume that $T \in \mathcal{T}_3$ admits a set of points P satisfying the lemma and let $z \in P$ correspond to a leaf of T. We show how to add two points x and y to T, so that $G(P \cup \{x, y\}, \beta')$ is a drawing of $T \cup \{(x, z), (y, z)\}$ and $G(P \cup \{x, y\}, \beta') = G(P \cup \{x, y\}, \beta) = G[P \cup \{x, y\}, \beta]$ for all β such that $\beta' \leq \beta \leq \infty$. In what follows, z' is the unique neighbor of z in T. To add a single point x, instead of a pair of points, the same method can be used.

The following two properties will be used in determining the positions of x and y.

Property 7 *There exists a disk D_1 centered at z such that, for every point $p \in D_1$ and every $v \in P - \{z\}$, if $R(z, v, \beta')$ contains a point of $(P - \{z, v\})$, then so does $R(p, v, \beta')$.*

Property 8 *There exists a disk D_2 centered at z such that, for every point $p \in D_2$ and every $\{u, v\} \subset P - \{z\}$, if $z \notin R[u, v, \infty]$, then $p \notin R[u, v, \infty]$.*

The first property ensures that if x and y are chosen from within D_1, then, for all β such that $\beta' \leq \beta \leq \infty$, the edge sets of $G(P \cup \{x, y\}, \beta)$ and $G[P \cup \{x, y\}, \beta]$ will be subsets of $T \cup \{(x, y), (x, z), (y, z), (x, z'), (y, z')\}$. The second property ensures that if x and y are chosen from within

D_2, then for all β such that $\beta' \leq \beta \leq \infty$, the edge sets of $G(P \cup \{x, y\}, \beta)$ and $G[P \cup \{x, y\}, \beta]$ will contain T.

Let $D = D_1 \cap D_2$. We will choose x and y from within D. All that remains is to show that x and y can be positioned within D so that the edges $\{(x, z), (y, z)\}$ are in both $G(P \cup \{x, y\}, \beta)$ and $G[P \cup \{x, y\}, \beta]$ for each β such that $\beta' \leq \beta \leq \infty$, and the edges $\{(x, y), (x, z'), (y, z')\}$ are in neither. To guarantee that the edges $\{(x, z), (y, z)\}$ exist, it suffices to ensure that the closed strips $R[x, z, \infty]$ and $R[y, z, \infty]$ contain no point of P other than z. To guarantee that the other three edges mentioned do not exist, choose x and y so that $\angle xzy > \max\{\pi/2, \alpha(\beta')\}$, $\angle xzz' \geq 2\pi/3$, and $\angle yzz' \geq 2\pi/3$. To see that this is possible, consider rays r_1 and r_2 emanating from z such that the angle between r_1 and r_2 is greater than $\max\{\pi/2, \alpha(\beta')\}$, and both the angle between r_1 and (z, z') and that between r_2 and (z, z') are greater than $2\pi/3$. See Figure 8. Note that r_1 and r_2 can be rotated slightly about z so that the lines they determine contain no points of $P - \{z\}$ and that the angle constraints still hold. Now x and y can be positioned along r_1 and r_2 so that the strips $R[x, z, \infty]$ and $R[y, z, \infty]$ contain no point of P other than z. □

Theorem 5 *Class $\mathcal{T}_3$ is (both open and closed) β-drawable for all β such that $\frac{\sqrt{3}}{2} < \beta \leq \infty$. Furthermore, given a $T \in \mathcal{T}_3$ and a β such that $\frac{\sqrt{3}}{2} < \beta \leq \infty$, a stable β-drawing of T can be computed in linear time in the real RAM model. Class $\mathcal{T}_3$ is not β-drawable for any β such that $0 \leq \beta \leq \frac{\sqrt{3}}{2}$*

Proof: The first part of the theorem follows immediately from Lemma 6. The drawing procedure described in the Lemma requires at most linear time with a real RAM model, since the disks D_1 and D_2 can each be computed in linear time. To prove the last part consider the tree T of Figure 9.a consisting of two adjacent vertices of degree 3. To see that T is not $[\frac{\sqrt{3}}{2}]$-drawable, first note that if it were, then all angles between consecutive edges would have to equal $2\pi/3$. Now it suffices to note that, no matter what lengths the edges have, $R[z, u, \frac{\sqrt{3}}{2}]$ contains none of $\{x, y, v, w\}$, contradicting the fact that T does not contain the edge (z, u). By Lemma 1, since T has no $[\frac{\sqrt{3}}{2}]$-drawing, it can have no (β)-drawing (or $[\beta]$-drawing) for any $\beta \leq \frac{\sqrt{3}}{2}$. □

4.2 The Class $\mathcal{T}_4$

Lemma 7 *Let T be any tree in $\mathcal{T}_4$ and let β' and β'' be such that $1 < \beta' < \beta'' < \infty$. There exists a set P of points in the plane such that, for each β, $\beta' \leq \beta \leq \beta''$, $G(P, \beta)$ is a stable (β)-drawing of T and $G[P, \beta]$ is a stable $[\beta]$-drawing of T.*

Proof: This proof is similar to the proof of Lemma 6 and is by induction on the number of vertices of T. Observe that both a tree consisting of a single vertex and a tree consisting of a vertex of degree four and four leaves can be drawn satisfying 1 and 2. Given a tree $T \in \mathcal{T}_4$ which admits a set of points P satisfying 1 and 2, and a point $z \in P$ corresponding to a leaf of T, we show how to add three new points w, x and y to P so that for each β such that $\beta' \leq \beta \leq \beta''$, $G(P \cup \{x, y, w\}, \beta) = G[P \cup \{x, y, w\}, \beta]$ is a drawing of $T \cup \{(x, z), (y, z), (w, z)\}$. As in the proof of Lemma 6, we choose disks D_1 and D_2 centered at z such that

- For every point $p \in D_1$, and every $v \in P - \{z\}$, if $R(z, v, \beta')$ contains a point of $P - \{z, v\}$, so does $R(p, v, \beta')$ (see Figure 10.a); and

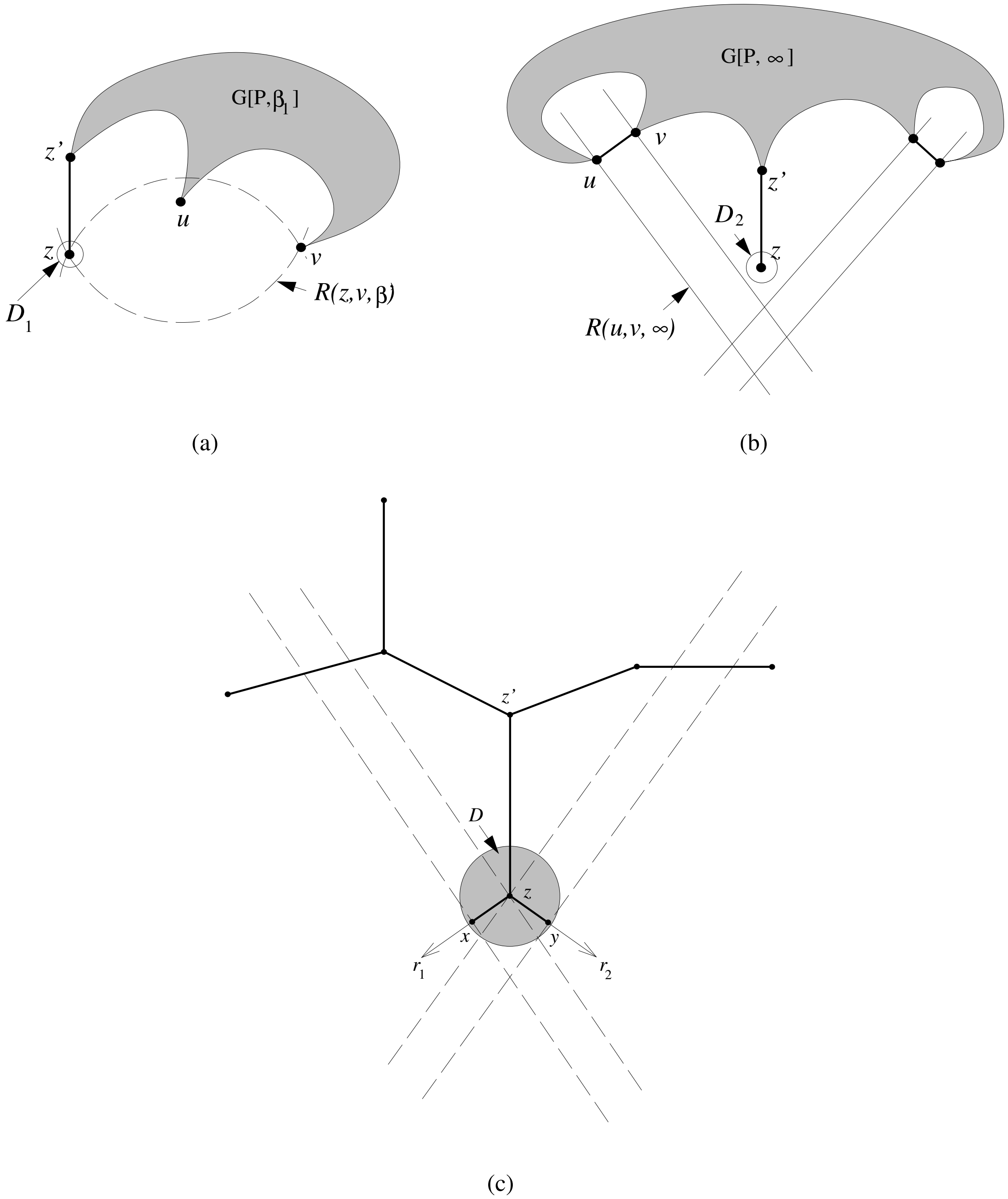

Figure 8: Illustration for Lemma 6.

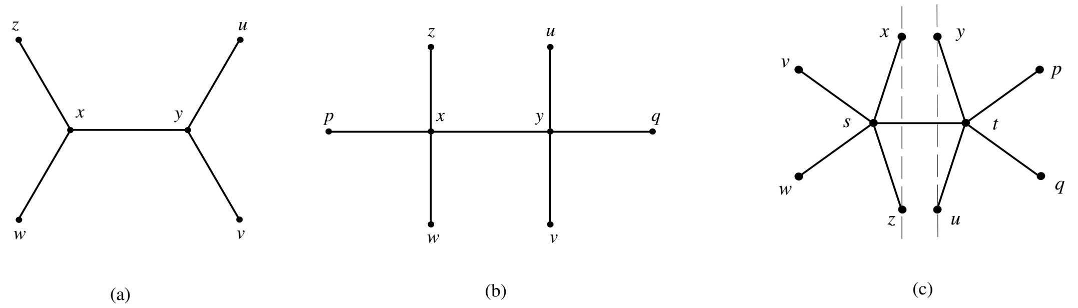

Figure 9: (a) A non-$\frac{\sqrt{3}}{2}$-drawable graph; (b) a non-(∞)-drawable graph; (c) a non-$(\frac{1}{\cos(\frac{2\pi}{5})})$-drawable graph.

- For every point $p \in D_2$, and every $u, v \in P - \{z\}$, if $z \notin R[u, v, \beta'']$, then $p \notin R[u, v, \beta'']$ (see Figure 10.b).

This guarantees that for any points $x, y, w \in D = D_1 \cap D_2$ and for any β such that $\beta' \leq \beta \leq \beta''$, the edge sets of $G(P \cup \{x, y, w\}, \beta)$ and $G[P \cup \{x, y, w\}, \beta]$ will contain T and contain no edges with one point in $\{x, y, w\}$ and the other in $P - \{z, z'\}$. By placing x, y and w in D so that the angles $\angle z'zy$, $\angle yzw$, $\angle wzx$, $\angle wzz'$ are all right angles, the only edges containing x, y, or w will be (x, z), (y, z) and (w, z). Thus $G(P \cup \{x, y, z\}, \beta)$ and $G[P \cup \{x, y, w\}, \beta]$ are both drawings of $T \cup \{(x, z), (y, z), (w, z)\}$. Again, as in Lemma 6, this technique will also work if only one or two vertices are to be attached to z. □

Theorem 6 *Class $\mathcal{T}_4$ is β-drawable for all β such that $1 < \beta < \infty$. Furthermore, given a $T \in \mathcal{T}_4$ and a β such that $1 < \beta < \infty$, a β-drawing of T can be computed in linear time in the real RAM model. Class $\mathcal{T}_4$ is not β-drawable for any other value of β.*

Proof: The first statement follows from Lemma 7; the second from the observation that disks D_1 and D_2 as in the proof of the lemma can be found in linear time. To prove the last statement, consider the tree T of Figure 9.b consisting of two adjacent vertices of degree 4. The characterization of Gabriel trees from [2] shows that T is not [1]-drawable. Therefore, by Lemma 1, T has no (β)-drawing (or $[\beta]$-drawing) for any $\beta \leq 1$. T has no $[\infty]$-drawing, since by Theorem 3, the minimum angle between any two consecutive edges must be greater than $\pi/2$. Finally, to see that T has no (∞)-drawing, first note that all angles between consecutive edges would have to equal $\pi/2$ and therefore, since $R(z, u, \infty) = R(x, y, \infty)$, T would have to have either both of (z, u) and (x, y) as edges, or neither. □

4.3 The Class $\mathcal{T}_5$

The range of values of β for which $\mathcal{T}_5$ is β-drawable is as yet unknown. In [2], it is shown that $\mathcal{T}_5$ is both (2)- and [2]-drawable. As immediate consequences of Theorem 3, no trees having any vertices of degree 5 can be β-drawn for any $\beta < \frac{1}{1-\cos(\frac{2\pi}{5})}$ or for any $\beta > \frac{1}{\cos(\frac{2\pi}{5})}$.

If we restrict our attention to either open or closed stable β-drawings of trees, then it is possible to prove that there exist trees in $\mathcal{T}_5$ that do not admit a stable β-drawable for $\frac{1}{1-\cos(\frac{2\pi}{5})} < \beta < 2$. The proof

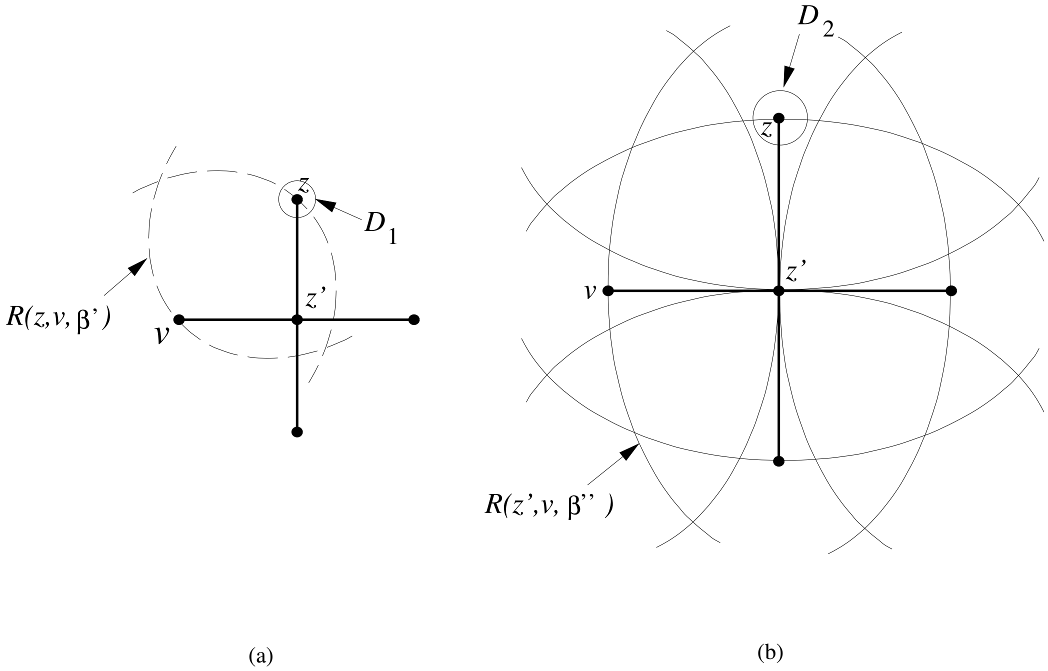

Figure 10: Illustration for Lemma 7.

is based on showing that there always exists a tree whose stable β-drawing would violate Theorem 3. We start with a geometric lemma.

Lemma 8 *Let $G(P,\beta)$ be a stable (β)-drawing of a tree and let $\frac{1}{1-\cos(\frac{2\pi}{5})} < \beta < 2$. Assume $G(P,\beta)$ contains a path $\Pi = xuvy$ such that $\angle xuv < \pi/2$ and x and y lie on the same half plane defined by the line through u and v. Then $\angle yvu > \pi/2 - \angle xuv + \alpha(\beta)$.*

Proof: Since $G(P,\beta)$ is stable, the region $R(x,y,\beta)$ contains at least one of the two vertices u and v. Suppose that $u \in R(x,y,\beta)$ (see Figure 11.a); this implies that $\angle xuy > \alpha(\beta)$ and that $\angle yuv < \angle xuv - \alpha(\beta)$. Also, since $y \notin R(u,v,\beta)$ and $\beta > 1$, point y is outside the disk $R(u,v,1)$, which implies $\angle yvu \geq \pi/2 - \angle yuv > \pi/2 - \angle xuv + \alpha(\beta)$. Consider now the case that $v \in R(x,y,\beta)$ (see Figure 11.b); this implies that $\angle yvx > \alpha(\beta)$. Also, since $x \notin R(u,v,\beta)$ and $\beta > 1$, point x is outside the disk $R(u,v,1)$, which implies $\angle xvu \geq \pi/2 - \angle xuv$. It follows that $\angle yvu = \angle yvx + \angle xvu > \pi/2 - \angle xuv + \alpha(\beta)$. □

With an analogous proof, we can prove the same result for $[\beta]$-stable drawings.

Lemma 9 *Let $G[P,\beta]$ be a stable $[\beta]$-drawing of a tree and let $\frac{1}{1-\cos(\frac{2\pi}{5})} < \beta < 2$. Assume $G[P,\beta]$ contains a path $\Pi = xuvy$ such that $\angle xuv < \pi/2$ and x and y lie on the same half plane defined by the line through u and v. Then $\angle yvu > \pi/2 - \angle xuv + \alpha(\beta)$.*

Theorem 7 *There always exists a tree $T \in \mathcal{T}_5$ which admits no stable β-drawing for $\frac{1}{1-\cos(\frac{2\pi}{5})} < \beta < 2$.*

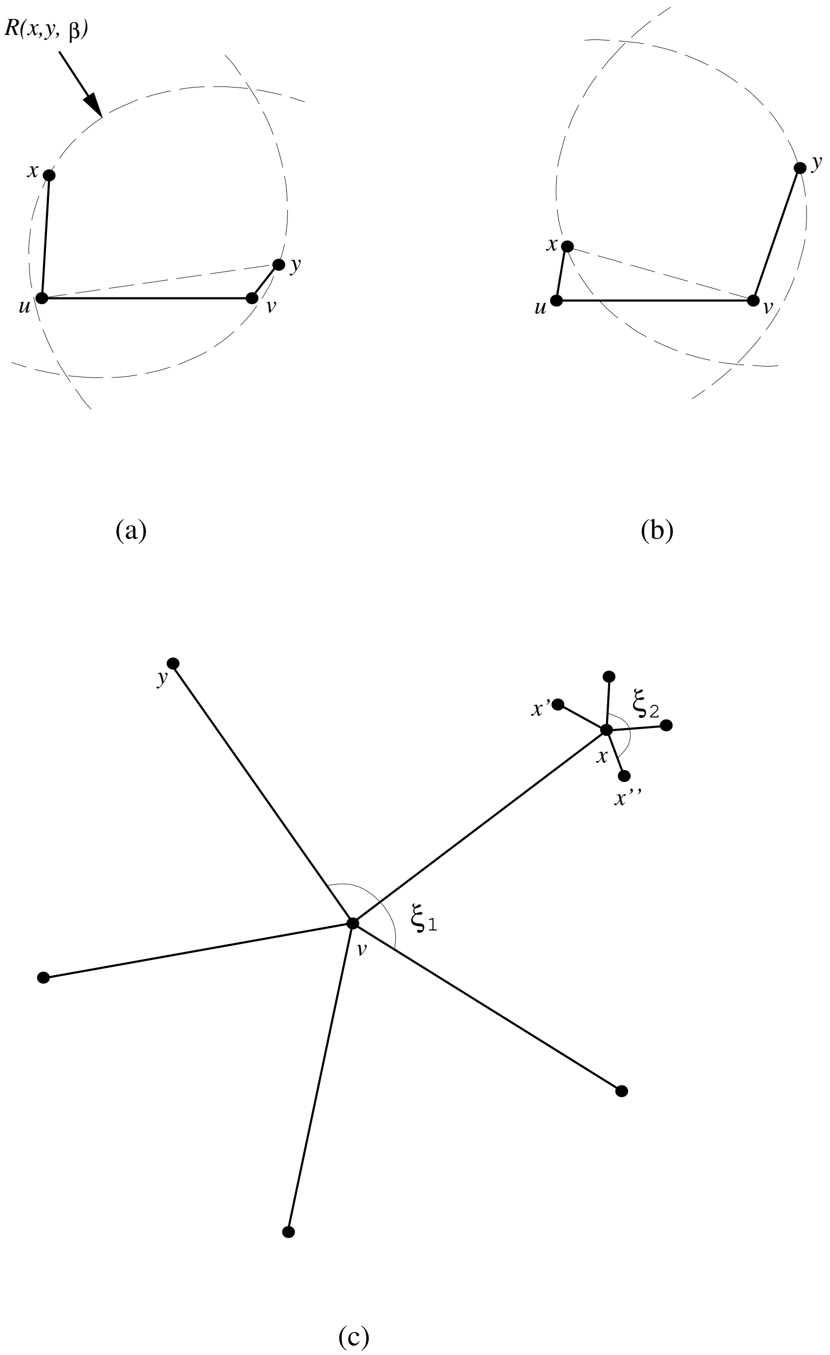

Figure 11: Illustration for Lemma 8 and for Theorem 7.

Proof: We first show how to construct a tree $T \in \mathcal{T}_5$ that does not admit a stable (β)-drawing for $\frac{1}{1-\cos(\frac{2\pi}{5})} < \beta < 2$. Then we prove that T has no stable $[\beta]$-drawing.

We begin by letting T consist of a vertex v of degree five and its neighbors. Consider any stable (β)-drawing of T. Let (y, v), (x, v), and (z, v) three consecutive edges encountered in this order when going clockwise around v (see Figure 11.c), such that the angle $\xi_1 = \angle yvz$ is $\xi_1 < \frac{2\pi - \alpha(\beta)}{2}$. That such three edges exist is a consequence of the fact that the minimum angle between any two consecutive edges is greater than $\alpha(\beta)$ (Theorem 3). Since $\beta < 2$, by Property 2 $\alpha(\beta) > \frac{\pi}{3}$ and both $\angle yvx$ and $\angle xvz$ are less than $\pi/2$.

We now add to T four new vertices each of which is a neighbor of x. Let x' and x'' be the two neighbors of x such that in the clockwise ordering of the vertices around x they appear as x'', v, x'. Since $\alpha(\beta) > \frac{\pi}{3}$ and $deg(x) = 5$, x' and x'' must lie in opposite half-planes with respect to the line through v and x in any stable (β)-drawing of T. Therefore, we can apply Lemma 8 to both path $x'xvy$ and path $x''xvz$ and obtain the following inequalities.

1. $\angle x'xv > \pi/2 - \angle yvx + \alpha(\beta)$, and
2. $\angle x''xv > \pi/2 - \angle zvx + \alpha(\beta)$.

Consider now the vertices adjacent to x. There exists an angle ξ_2 defined by three consecutive edges incident on x such that

$$\xi_2 < 2\pi - \angle x'xv - \angle x''xv - \alpha(\beta) < 2\pi - \pi + \angle yvx + \angle zvx - 2\alpha(\beta) - \alpha(\beta) = \xi_1 - (3\alpha(\beta) - \pi).$$

Since $\alpha(\beta) > \frac{\pi}{3}$, we have that $\xi_1 - \xi_2 > 3\alpha(\beta) - \pi$, which is a positive constant for a given value of β. By iterating this process of adding degree five vertices to T, we can obtain an arbitrarily long sequence of angles $\xi_1, \xi_2, \ldots, \xi_m$ such that for each $k \leq m$, $\xi_{k-1} - \xi_k > 3\alpha(\beta) - \pi$. Thus, eventually ξ_m becomes negative, a contradiction.

The proof is now completed by observing that inequalities 1 and 2 can be also obtained by applying Lemma 9 to a stable $[\beta]$-drawing of T; thus if T does not admit a stable (β)-drawing, it does not admit a stable $[\beta]$-drawing either. □

5 Characterizing Representable Trees

In this section we prove the correctness of the claims in Table 1. Much of the content of the Table is based on what we have presented in the previous sections, but other results are still necessary. We thus begin by characterizing those trees which admit a $[\infty]$-drawing and those which admit a $[\frac{\sqrt{3}}{2}]$-drawing.

Theorem 8 *A tree T is $[\infty]$-drawable if and only if $T \in \mathcal{T}_3$.*

Proof: By Theorem 5, every tree in $\mathcal{T}_3$ has a $[\infty]$-drawing. To show that these are the only trees that have such drawings, it suffices to observe that, because of Theorem 3, the angle between consecutive edges in a $[\infty]$-drawing of a tree must be greater than $\pi/2$; thus there can be no vertices of degree greater than 3. □

The characterization of $[\frac{\sqrt{3}}{2}]$-drawable trees requires some more effort. We first give an upper bound to the maximum vertex degree of such trees and then we study the geometric properties of $[\frac{\sqrt{3}}{2}]$-drawings. Theorem 3 and Property 2 imply the following.

Lemma 10 *A $[\frac{\sqrt{3}}{2}]$-drawable tree cannot have a vertex of degree greater than three.*

To complete the characterization, we exploit the relationship between the Delaunay triangulation and $\beta - drawings$, as described in Section 3.

Lemma 11 *If $G[P, \frac{\sqrt{3}}{2}]$ is a tree and if (a, b) and (b, c) are two edges of $G[P, \frac{\sqrt{3}}{2}]$ such that $\angle abc = \frac{2\pi}{3}$, then $\triangle abc$ is in $DT(P)$.*

Proof: First observe that $G[P, \frac{\sqrt{3}}{2}]$ is a subgraph of $DT(P)$ since $G[P, \frac{\sqrt{3}}{2}] = G[P, 1]$ by Lemma 1 and since $G[P, 1] \subseteq DT(P)$ by Property 6. Thus (a, b) and (b, c) are in $DT(P)$, so we need only show that $DT(P)$ also contains (a, c). If (v, a) and (v, b) are two edges in any triangulation of P, then there exists a path from a to b which does not contain v and such that every vertex on the path is adjacent to v. Hence, $DT(P)$ contains a path Π from a to c not including b such that every edge on that path is adjacent to b. Π, along with edges (a, b) and (b, c) form a cycle. By Theorem 1, some edge (x, y) on Π must be such that $\angle xby \geq \frac{2\pi}{3}$. But $\angle abc = \frac{2\pi}{3}$, thus (x, y) is the edge (a, c) and so $\triangle abc$ is in $DT(P)$. $\square$

Corollary 2 *A $[\frac{\sqrt{3}}{2}]$-drawable tree cannot have two adjacent vertices of degree three.*

Proof: (By contradiction). Suppose, that there were a $[\frac{\sqrt{3}}{2}]$-drawable tree T with two adjacent vertices of degree three (see Figure 9). Then, in any $[\frac{\sqrt{3}}{2}]$-drawing of T there would be a path $\Pi = zxyu$ such that $\angle zxy = \frac{2\pi}{3}$ and $\angle xyu = \frac{2\pi}{3}$. Because of Lemma 11, the edge (z, y) is in $DT(P)$ as is the edge (x, u). This, however, contradicts the planarity of $DT(P)$. $\square$

The characterization is now completed by showing that all trees with maximum vertex degree at most three and no two adjacent vertices of degree three are $[\frac{\sqrt{3}}{2}]$-drawable. We exhibit a drawing algorithm based on the following lemma.

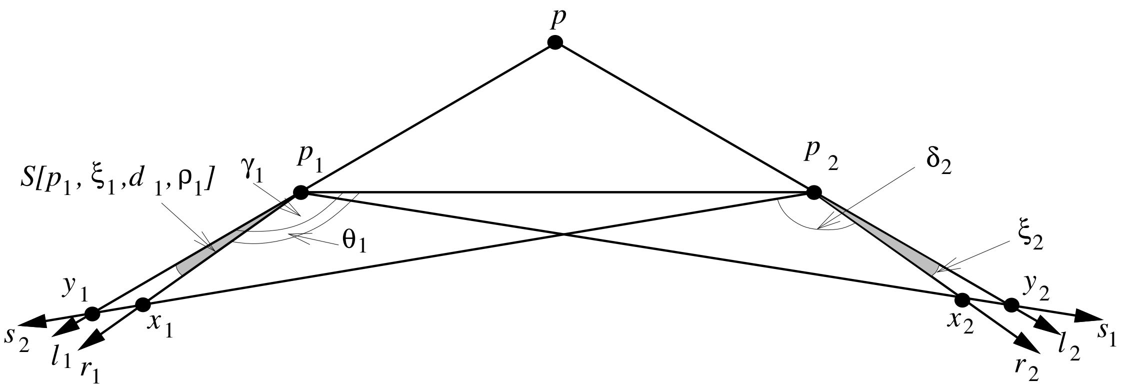

Figure 12: Illustration for Lemma 12.

Lemma 12 *Let $P = \{p, p_1, p_2\}$ be a set of points in the plane such that $\angle p_1pp_2 = \frac{2\pi}{3}$ and $d(p, p_1) = d(p, p_2)$. There exist angles $\xi_1 > 0$, $\xi_2 > 0$, vectors d_1, d_2, and radii $\rho_1 > 0$, $\rho_2 > 0$ such that P separates $S_1 = S[p_1, \xi_1, d_1, \rho_1]$ and $S_2 = S[p_2, \xi_2, d_2, \rho_2]$.*

Proof: See Figure 12. Let l_i, r_i, and s_i be three vectors emanating from p_i $(i = 1, 2)$ external to $\triangle pp_1p_2$ such that

1. l_i is collinear with $\overline{pp_i}$, forming an angle γ_i with $\overline{p_1p_2}$.
2. r_i forms an angle θ_i with $\overline{p_1p_2}$ such that $\frac{2\pi}{3} < \theta_i < \gamma_i$.
3. s_i and r_i form an angle δ_i, such that $\theta_i > \delta_i > \frac{2\pi}{3}$.

Let $\xi_i = \gamma_i - \theta_i > 0$. Choose d_i as the bisector of angle ξ_i. Denote the points at which s_1 intersects r_2 and l_2 by x_2 and y_2 respectively. Similarly, denote by x_1 and y_1 the points at which s_2 intersects r_1 and l_1. Choose ρ_i such that sector $S_i = S[p_i, \xi_1, d_1, \rho_1]$ is inside $\triangle p_ix_iy_i$. Let u be any point in S_1 and v be any point in S_2; since $\angle(up_1v) \geq \delta_1 > \frac{2\pi}{3}$ and $\angle(vp_2u) \geq \delta_2 > \frac{2\pi}{3}$, we can conclude that P separates S_1 and S_2. $\square$

Lemma 13 *Let $T \in \mathcal{T}_3$ be such that no two adjacent vertices of T have degree three, and let u be a leaf of T. (T, u) can be $[\frac{\sqrt{3}}{2}]$-S_ϕ-drawn for any $\phi > 0$.*

Proof: Since $T \in \mathcal{T}_3$, it has maximum vertex degree at most three. Given a point q in the plane, angle $\phi > 0$, a vector d and a radius $\rho > 0$, procedure $Draw[\frac{\sqrt{3}}{2}]$ (see Figure 13), will produce a $[\frac{\sqrt{3}}{2}]$-drawing of (T, u) in $S[q, \phi, d, \rho]$.

The proof of correctness is by induction on the size of T. We adopt the notation of Figure 13. Observe first that $Draw[\frac{\sqrt{3}}{2}]$ is only invoked on subtrees of (T, u) where $deg_T(u)$ is at most two. If T has a single vertex, $Draw[\frac{\sqrt{3}}{2}]$ produces the correct type of drawing. Suppose that the procedure works correctly on all trees with maximum vertex degree at most three, no two adjacent degree three vertices, and having fewer than n vertices. Assume now that $Draw[\frac{\sqrt{3}}{2}]$ is invoked on a subtree (T_0, u_0) of (T, u) having n vertices and let v be the neighborhood of u_0. There are two cases to consider.

If $deg_T(v) = 1$, then a sector $S[p, \phi, d, \rho']$ is found which lies in the interior of $S[q, \phi, d, \rho]$. By induction, $(T_v(u_0), v)$ will be drawn inside this smaller sector, guaranteeing an S_ϕ-drawing of (T, u) in $S[q, \phi, d, \rho]$.

If $deg_T(v) = 2$, let v_1 and v_2 denote the neighbors of v other than u. Since T does not have two adjacent vertices of degree three both v_1 and v_2 have degree two in T. Now, points $P = \{p, p_1, p_2\}$, are selected from the interior of $S[q, \phi, d, \rho]$ so as to satisfy the hypotheses of Lemma 12. This allows sectors $S_i = S[p, \xi_i, d_i, \rho]$ to be found such that both S_1 and S_2 are in the interior of $S[q, \phi, d, \rho]$ and such that $\{p\}$ separates S_1 and S_2. Each subtree $(T_{v_i}(v), v_i)$, $i = 1, 2$ is then recursively drawn correctly in the sector S_i. $\square$

Observe that the drawing procedure of the preceding lemma takes linear time using the real RAM model of computation, and produces a stable drawing of the input. The results of Corollary 2 and Lemma 13 are summarized in the following theorem.

Theorem 9 *A tree is $[\frac{\sqrt{3}}{2}]$-drawable if and only if it has maximum vertex degree at most three and no two adjacent vertices have degree three. Furthermore, in the real RAM model, a stable $[\frac{\sqrt{3}}{2}]$-drawing of a representable tree can be computed in time proportional to the size of a tree.*

procedure $Draw[\frac{\sqrt{3}}{2}](T, u, \phi, q, d, \rho)$

Draw u at q.

Let v denote the neighbor of u.

Case: $deg_T(v) = 1$.

Choose $\rho' < \rho$ so that for $p = q + (\rho - \rho')d$, $S[p, \xi, d, \rho']$ lies in the interior of $S[q, \phi, d, \rho]$.

Draw a segment from q to q'.

$Draw[\frac{\sqrt{3}}{2}](T_v(u), v, \phi, p, d, \rho')$

Case: $deg_T(v) = 2$.

Let v_1, v_2 denote the neighbours of v other than u.

Choose $\rho' < \rho$ so that $p = q + (\rho - \rho')d$ lies in the interior of $S[q, \phi, d, \rho]$.

Draw a segment from q to p.

Draw v at p.

Choose p_1, p_2 in the interior of $S[q, \phi, d, \rho]$, so that $\angle p_1 p p_2 = \frac{2\pi}{3}$ and $d(p, p_1) = d(p, p_2)$.

For $i = 1, 2$ draw a segment from p to p_i.

Let ξ_i, d_i, ρ_i be chosen as described in Lemma 12 and such that $S_i = S[p_i, \xi_i, d_i, \rho_i]$ is contained in the interior of sector $S[q, \phi, d, \rho]$.

For $i = 1, 2$: $Draw[\frac{\sqrt{3}}{2}](T_{v_i}(v), v_i, \xi_i, p_i, d_i, \rho_i)$.

end Procedure.

Figure 13: Procedure $Draw[\frac{\sqrt{3}}{2}](T, u, \phi, q, d, \rho)$.

We are now in a position to verify Table 1.

Theorem 10 *As β ranges from 0 to ∞, the sets $\mathcal{T}(\beta)$ and $\mathcal{T}[\beta]$ change as shown in Table 1.*

Proof: There are twenty-two statements to be proved. We refer to statements in the Table as follows: $\mathcal{S}(k)$ refers to the statement of row k and column $\mathcal{T}(\beta)$ new; similarly, $\mathcal{S}[k]$ refers to the statement of row k and column $\mathcal{T}[\beta]$ new. We prove the statements in order of decreasing value of k. Recall that by Corollary 1, we need not consider trees containing vertices of degree greater than five.

The proof of statement $\mathcal{S}[13]$ is just Theorem 8. To show $\mathcal{S}(13)$, observe first that by Theorem 5, $\mathcal{T}_3 \subset \mathcal{T}(\infty)$. Now note that the tree consisting of a single vertex of degree 4 and its neighbors is (∞)-drawable and the tree in Figure 9.b is not (see Theorem 6).

By Theorem 6, both $\mathcal{T}(\beta)$ and $\mathcal{T}[\beta]$ contain $\mathcal{T}_4$ for the values of β in rows 6 to 12 of the Table. Thus, statements $\mathcal{S}[12]$ and $\mathcal{S}[11]$ hold by Lemma 5 and statement $\mathcal{S}(12)$ holds by Lemma 4.

To verify the containments in $\mathcal{S}(11)$, first observe that the tree consisting of a single vertex of degree 5 and its neighbors is $(\frac{1}{\cos(\frac{2\pi}{5})})$-drawable. Now consider the tree T of Figure 9.c consisting of two adjacent vertices of degree 5. To see that T is not $[\frac{1}{\cos(\frac{2\pi}{5})}]$-drawable, first note that if it were, then all angles between consecutive edges would have to equal $2\pi/5$ and all edges would have to have the same length. Now it suffices to note that since $R(x, y, \infty)$ contains no other vertices of T, neither does $R(x, y, \frac{1}{\cos(\frac{2\pi}{5})})$. This contradicts the fact that T does not contain the edge (x, y).

To verify statements $\mathcal{S}[10]$ to $\mathcal{S}(8)$, it suffices to observe that the tree consisting of a single vertex of degree 5 and its neighbors is both (β)- and $[\beta]$-drawable for these ranges of β.

The proof of statement $\mathcal{S}[7]$ is similar to that of $\mathcal{S}(11)$; statements $\mathcal{S}(7)$ down to $\mathcal{S}(6)$ follow from Lemma 4.

Lemma 1, along with the results on $\mathcal{T}(1)$ of [2], imply the truth statements $\mathcal{S}(4)$ and $\mathcal{S}[4]$. Statement $\mathcal{S}[3]$ is just Corollary 2.

Theorem 4 and Theorem 3 directly imply statements $\mathcal{S}(3)$ down to $\mathcal{S}[1]$. Finally, statement $\mathcal{S}(1)$ can be proved by observing that every graph which is (0)-drawable is a clique, and the only trees which are cliques are the trees on 1 and 2 vertices. □

6 A Characterization of $G[P, \infty]$

We begin by describing a relation between $G[P, \infty]$ and minimum spanning trees of P.

Lemma 14 *For each finite set P of points in the plane, $G[P, \infty]$ is a subgraph of the intersection of all minimum spanning trees of P. Thus $G[P, \infty]$ is a forest.*

Proof: Consider an edge (x, y) of $G[P, \infty]$. Suppose there exists a minimum spanning tree T of $G[P, \infty]$ not containing (x, y). Consider the path in T between x and y. Some edge (u, v) of the path must cross $R[x, y, \infty]$, but neither u nor v are in $R[x, y, \infty]$. This implies that (u, v) is longer than (x, y), a contradiction. □

An interesting consequence of Lemma 14 is the following:

Corollary 3 *For every P, $G[P, \infty]$ is a stable drawing.*

Proof: Consider a non-edge (u, v) of an $[\infty]$-drawing of a tree T. Every edge on the path between u and v in T is shorter than $d(u, v)$, thus some point on the path must be in $R[u, v, \infty]$. □

Theorem 11 *A graph G is $[\infty]$-drawable if and only if every connected component of G is in $\mathcal{T}_3$ and G is not one of the following graphs: two non-adjacent vertices, a vertex and a non-adjacent edge, or a pair of non-adjacent edges.*

Proof: By Theorem 3, G can have no vertices of degree greater than three. Also, it can easily be checked that none of the three graphs mentioned in the theorem are $[\infty]$-drawable. All that remains is to show that any other forest each of whose components is in $\mathcal{T}_3$ is $[\infty]$-drawable. We do this by describing two constructions for creating $[\infty]$-drawings: the first will be used when the forest has at least three components, the second when the forest has exactly two components. If there is only one component, the construction of Lemma 6 can be used.

Suppose that T has at least $k \geq 3$ components. Choose a set C of k points such that for each pair $x, y \in C$, $R[x, y, \infty]$ contains some point $z \in C - \{x, y\}$ in its interior (for example, one can suitably choose C as a subset of the vertices of a triangular grid). For each $p \in C$, one can define a disk D_p having center

p and with the following property: For each pair of points $x, y \in C$, there exists a $z \in C - \{x, y\}$ such that for every $x' \in D_x$ and $y' \in D_y$, $R[x', y', \infty]$ contains D_z. Using the construction of Lemma 6, the components of T can be drawn one in each of the disks D and there will be no edges between components. By correct positioning and scaling of the drawing of each component, it can be guaranteed that no vertex of any component lies in the infinite strip of an edge in any other component. See Figure 14.a.

Now assume that T has exactly two components, T' and T'', such that, without loss of generality, T' contains at least two edges. Use the construction of Lemma 6 to produce a $[\infty]$-drawing of T', let z be a leaf of T' and let z' be the neighbor of z. As in the proof of the Lemma, there exist disks D_1 and D_2 centered at z having Properties 7 and 8. Let $D = D_1 \cap D_2$. If there exists a non-neighbor v of z such that v is not collinear with the edge (z, z'), then there exists a disk $D' \subset D$ such that no point of D' is in $R[z, z', \infty]$ and for every point $p \in D'$, $R[p, z, \infty]$ contains v (see Figure 14.b). It is now possible to draw T'' inside D', obtaining a $[\infty]$-drawing of T. The existence of such a v can be guaranteed by a slight modification to the construction in Lemma 6. □

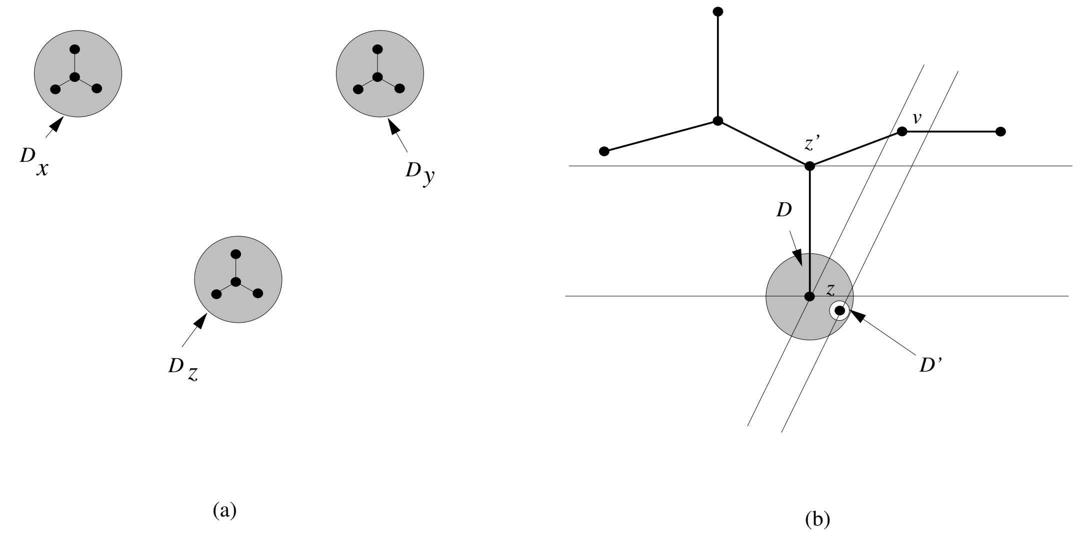

Figure 14: How to draw closed strip drawable graphs.

7 Conclusions and Open Problems

This paper provides characterizations of which trees are β-drawable, for infinitely many values of β. These characterizations give rise to linear time recognition and drawing algorithms for each type of proximity tree. Furthermore, a complete characterization of all proximity drawable graphs when the proximity region is the closed infinite strip is given. To date, this is the first complete characterization of proximity drawable graphs.

For $\frac{1}{1-cos(\frac{2\pi}{5})} \leq \beta \leq \frac{1}{cos(\frac{2\pi}{5})}$ β-trees are only partially characterized. In this interval, we prove that all trees with maximum vertex degree at most four are both (β)- and $[\beta]$-drawable. In [2] it is shown that all trees of class T_5 admit both (β)- and $[\beta]$-drawings when $\beta = 2$. We provide a negative result for the stable β-drawability of $\mathcal{T}_5$ when $\frac{1}{1-cos(\frac{2\pi}{5})} < \beta < 2$. It would be interesting to close the gap by answering the following question.

Question 1 *For $\beta \neq 2$ such that $\frac{1}{1-cos(\frac{2\pi}{5})} \leq \beta \leq \frac{1}{cos(\frac{2\pi}{5})}$, which trees are β-drawable?.*

We have used β-stability as a basic tool for both characterizing and drawing graphs. Observe that all the β-drawings produced by the algorithms of this paper are stable. Considering the results presented in Subsection 4.3, a question whose answer might be very helpful in closing the gap is the following.

Question 2 *Does every β-tree admit a stable β-drawing?.*

To date, little work has been done on the problem of characterizing families of β-drawable graphs other than trees. Lubiw and Sleumer [16] show that maximal outerplanar graphs admit both [1]-drawings (Gabriel drawings) and (2)-drawings (relative neighborhood drawings). They also prove that all biconnected outerplanar graphs are (2)-drawable. It is easy to see that a triangulated planar graph is not β-drawable for $\beta > 2$. However, for smaller values of β one is led to ask:

Question 3 *For $\beta \leq 2$, which triangulated planar graphs are β-drawable?.*

References

[1] J. A. Bondy and U. S. R. Murty. *Graph Theory with Applications.* Elsevier Science, New York, New York, 1976.

[2] P. Bose, W. Lenhart, and G. Liotta. Characterizing Proximity Trees. *Algorithmica*, Special Issue on Graph Drawing (to appear). Also available as TR-SOCS 93.9, School of Computer Science, McGill University, 1993.

[3] R. J. Cimikowski. Properties of Some Euclidean Proximity Graphs. *Pattern Recognition Letters,* **13**, 1992, pp. 417-423.

[4] G. Di Battista, P. Eades, R. Tamassia and I.G. Tollis. Algorithms for Automatic Graph Drawing: An Annotated Bibliography. *Computational Geometry: Theory and Applications* **4**, 1994, pp. 235-282.

[5] G. Di Battista, W. Lenhart, and G. Liotta. Proximity Drawability: a Survey. *Graph Drawing, Proc. of the DIMACS International Workshop GD94, New Jersey, USA, October 1994*, Lecture Notes in Computer Science LNCS **894**, R. Tamassia and I.G. Tollis *eds.*, Springer-Verlag, 1995, pp. 328-339.

[6] G. Di Battista and L. Vismara. Angles of Planar Triangular Graphs. *Proceedings STOC '93,* 1993, pp. 431-437.

[7] P. Eades, T. Lin, and X. Lin. Two Tree Drawing Conventions. *International Journal of Computational Geometry and Applications,* **3**, 1993, pp. 133-153.

[8] P. Eades. Drawing Free Trees. *Bullettin of the Institute of Combinatorics and its Applications,* 1992, pp. 10-36.

[9] P. Eades and S. Whitesides. The Realization Problem for Euclidean Minimum Spanning Tree is NP-hard. *Proceedings 10th ACM Symposium on Computational Geometry,* pp. 49-56, 1994.

[10] H. ElGindy, G. Liotta, A. Lubiw, H. Meijer, S.H. Whitesides. Recognizing Rectangle of Influence Drawable Graphs. *Graph Drawing, Proc. of the DIMACS International Workshop GD94, New Jersey, USA, October 1994,* Lecture Notes in Computer Science LNCS **894**, R. Tamassia and I.G. Tollis *eds.*, Springer-Verlag, 1995, pp. 352-363.

[11] I. Fary. On Straight Lines Representation of Planar Graphs. *Acta Sci. Math. Szeged,* **11**, 1948, pp. 229-233.

[12] H. de Fraysseix, J. Pach, and R. Pollack. Small Sets Supporting Fary Embbeddings of Planar Graphs. *Proceedings ACM Symposium on Theory of Computing '88,* 1988, pp. 426-433.

[13] H. de Fraysseix, J. Pach, and R. Pollack. How to Draw a Planar Graph on a Grid. *Combinatorica,* **10**, 1990, pp. 41-51.

[14] G. Kant. Drawing Planar Graphs Using the *lmc*-ordering. *Proceedings IEEE Symposium on Foundations of Computer Science,* 1992, pp. 101-110.

[15] D. G. Kirkpatrick and J. D. Radke. A Framework for Computational Morphology. *Computational Geometry,* ed. G. T. Toussaint, Elsevier, Amsterdam, 1985, pp. 217-248.

[16] A. Lubiw, and N. Sleumer. All Maximal Outerplanar Graphs are Relative Neighborhood Graphs. *Proceedings fifth Canadian Conference on Computational Geometry,* Waterloo, 1993, pp. 198-203.

[17] S. Malitz and A. Papakostas. On the Angular Resolution of Planar Graphs. *Proceedings ACM Symposium on Theory of Computing,* 1992, pp. 527-538.

[18] D. W. Matula and R. R. Sokal. Properties of Gabriel Graphs Relevant to Geographic Variation Research and the Clustering of Points in the Plane. *Geographical Analysis,* **12**, 1980, pp. 205-222.

[19] C. Monma and S. Suri. Transitions in Geometric Minimum Spanning Trees. *Proc. 7th ACM Symposium on Computational Geometry,* 1991, pp. 239-249.

[20] F. P. Preparata and M. I. Shamos. *Computational Geometry – an Introduction.* Springer-Verlag, New York, 1985.

[21] J. D. Radke. On the shape of a set of points. *Computational Morphology,* ed. G. T. Toussaint, Elsevier, Amsterdam, 1988, pp. 105-136.

[22] G. T. Toussaint. The Relative Neighborhood Graph of a Finite Planar Set. *Pattern Recognition,* **12**, 1980, pp. 261-268.

[23] W. T. Tutte. Convex Representations of Graphs. *Proceedings London Math. Society,* **10**, 1960, pp. 304-320.

[24] R. B. Urquhart. Some Properties of the Planar Euclidean Relative Neighborhood Graph. *Pattern Recognition Letters,* **1**, 1983, pp. 317-322.