ON DETERMINING THE GENUS OF A GRAPH IN $O(v^{O(g)})$ STEPS[†]
(Preliminary Report)

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1. INTRODUCTION

In this paper we present an algorithm which on input a graph G and a positive integer g finds an embedding of G on a surface on genius g, if such an embedding exists. This algorithm runs in $(v)^{O(g)}$ steps where v is the number of vertices of G.

As many other authors (Hefter, Edmonds) have noted finding a topological embedding can be reduced to a purely combinatorial problem. Namely, a topological embedding of G onto some 2-dimensional orientable surface determines a cyclic ordering of the edges at each vertex and any cyclic ordering of the edges at every vertex determines a class of embeddings of G. We shall, in the next section make the combinatorial notation more precise but leave the correspondence between topological and combinatorial embeddings to the reader.

We believe that removing the nondiscrete topological definitions (i.e., the notation or differentiability, 2-dimensional surface, etc.) from our formal definitions has a multitude of advantages. First our goal is to produce an algorithm which operates on discrete machines and thus at some point we must remove these notions anyway. Secondly, demonstrations on proofs in the amalgam of graph theory and topology have been riddled with flaws (e.g., 4-color theorem, planarity algorithms, Jordan curve theorem), and which, no doubt, this paper also suffers. The hope is that a combinatorial proof may transcend these problems. Third, our main goal is not just to draw graphs on "inner tubes" but to understand how graph theory, topology and computational complexity interact. We have kept no definitions sacred and we have redefined the notion of a graph. We have even rewritten Euler's formula.

2. NOTATIONS AND DEFINITIONS

2.1. Graphs

It will be convenient, for our purposes, to adopt a somewhat unorthodox definition of graphs.

The edges of a graph will be finite sets of points. An edge must contain at least one point in addition to its extremities. The extremities of an edge are called vertices. In this way our definition resembles closely that of a topological graph (cf Ma [67]), except that our

[†]This paper is the result of two different papers submitted to the Conferences: one by the first two authors and the other one by the third. The two papers being quite different in their approach, it was difficult, in the short time available, to produce a truly joint paper. As a result this paper mostly reflects the approach of the first paper. The details of both these approaches will appear elsewhere.

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edges are not open segments but ordered finite sets of points. Two points will be called related if they are neighbors on the same edge. Formally, we have the following

Definition. A graph G is a triple (P,V,R) where

(1) P is a finite set whose elements are called points;

(2) V is a subset of P whose elements are called vertices;

(3) R is an antireflexive and symmetric relation on P such that

(3.1) no two vertices are related;

(3.2) points in $P-V$ are related to at most two other points.

Let R' be the restriction of R to $P-V$. The reflexive and transitive closure of R' is an equivalence relation on $P-V$. Its equivalence classes will be called the (open) edges of G. From 3.2) it follows that one can linearly order (in exactly two different ways) the edges of G. There are at most two vertices related to the points of an edge e called the extremities of e. The set consisting of an edge e together with its extremities and will be denoted by $\bar{e}$ will be called a closed edge of G and will be denoted by $\bar{e}$. An edge e will be said to connect its extremities or to be related to them.

The (open) edge e with extremities u and v can thus receive two orientations which will be denoted by (uv) and (vu). Likewise, the orientations of the closed edges will be $[uv]$ and $[vu]$ respectively.

The pair (P,R) defines a simple (or simplicial) graph. A graph G with the property that every point $p \in P-V$ is related to exactly two other points will be called closed. A graph that is not closed will be called open.

A graph $G'=(P',V',R')$ is a subgraph of G if $P' \subset P, V' \subset V$ and R' is the restriction of R to P'.

A morphism (or simply a map) from G to G' (written $f:G \to G'$) is defined by giving a map $f:P \to P'$ such that (a) $f(v) \in V'$, (b) $f(P-V) \subset P'-V'$ and that (c) for any two points $p,q \in P$, pRq implies $f(p)Rf(q)$.

A morphism is surjective if f_p is surjective.

Let G' be a subgraph of G. The complement of G' in G, $G-G'$ is the graph $G''=(P'',V'',R'')$ where $P''=P-P'$, $V''=V \cap (P-P')$ and R'' is the restriction of R to P''. The complement of a closed subgraph is open. The smallest closed subgraph of G containing G' denoted $cl(G')$ is called the closure of G' in G. If G is closed, then $cl\,(G')$ is closed.

The reflective and transitive closure of R is an equivalence relation whose equivalence classes are called the components of G. If G' is a subgraph of a graph G, the components of $G-G'$ are called pieces. The attachments of a piece C of $G-G'$ are the points of G' related to some point of C. An attachment is also a point of $cl(C)-C$.

A closed graph G will be called reduced if no triple (V', P,R) with $V' \subsetneq V$ is a graph.

A chain C of a graph G is an alternating sequence of vertices and edges any two consecutive elements of which are related. If the first and the last elements of the sequence are vertices, the chain is called closed. The first and last vertex of a closed chain are its extremities. A closed chain is a closed subgraph of G. For any chain C, the closure of C will be denoted by $[C]$. The open part (C) of a chain C is the chain obtained by removing the extremities of $[C]$. A closed chain with the same extremities is a cycle.

2.2. Embedding of graphs.

An embedding I of a graph G is simply a cyclic orientation of the edges associated with each vertex of G.

A pair (G,I) consisting of a graph and an embedding will be called an embedded graph, often denoted G_I.

Each edge e of G has in a natural way two sides. Formally, if x and y are the two vertices to an edge e, then the triples (x,e,y) and $(y,e,x,)$ are the two sides of e. If e and e' are edges of x, then the triple (e,x,e') is a corner of x, sometimes denoted $\hat{x}$.

If G_I is an embedded graph, e and e' are two consecutive edges of x with respect to I then (e,x,e') is an elementary corner of x. A face of G_I is a minimal cycle of G_I such that consecutive triples are either sides or elementary corners. We shall let $\langle x_1,\ldots,x_n\rangle$ denote the cyclic order of $x_1,\ldots,x_n$ induced by the linear order $(x_1,\ldots,x_n)$. Thus, the faces of an embedded graph are simply cycles. We shall let f denote the number of faces of an embedded graph.

Note that this definition only agrees with the usual definition when the embeded graph G_I is a connected graph. It is fundamental to our approach to consider embeddings of disconnected graphs or which produce disconnected graphs.

As a simple example consider a graph consisting of two disjoint cycles. See Figure 1.

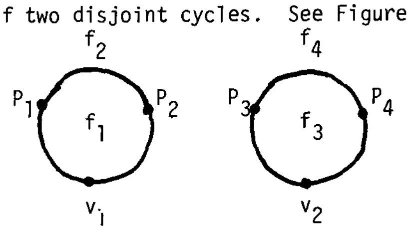


Figure 1.

The graph has 2 vertices, v_1 and v_2, and 2 edges, one containing the points p_1 and p_2 and the other containing the points p_3 and p_4. There is only one embedding of this graph since each vertex is of valence 2 and hence has only one orientation. This embedding has four faces, f_1, f_2, f_3 and f_4. Notice that the cycles $\langle v_1, (p_1,p_2)\rangle$ and $\langle v_2, (p_3,p_4)\rangle$ are two different faces f_2 and f_4 respectively.

In order to use Euler's formula, we must introduce a term for the number of connected components. If G_I is an embedded graph with v vertices, e edges, f faces, and p connected components, then the genus g of G_I satisfies the Euler-Poincaré Formula for graphs: $f - e + v = 2(p-g)$. This formula will be proved essentially when we review homology. We simply call this the E-P formula. The Euler Characteristic is $\chi(G_I) = f - e + v$.

We could either view the E-P formula as a theorem about the genus of G_I or as a definition of the genus of G_I. We shall formally do the latter and leave the correspondence with the topological notion to the reader.

Definition. The genus of G_I is the g satisfying the E-P formula.

2.3. Embeddings with Splittings

There are two natural operations on a graph which we will need, splitting vertices and identifying vertices. As an example consider the embedded graph as shown in Figure 2.

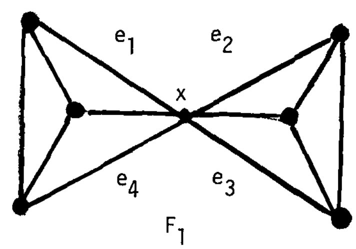


Figure 2.

The triples (e_1,x,e_2) and (e_3,x,e_4) are corners of the external face, say, F_1. In fact it is always the case that each cut-point of an embedded graph lies on the same face at least twice. A standard procedure is to partition this graph into 2-connected components -- see Figure 3.

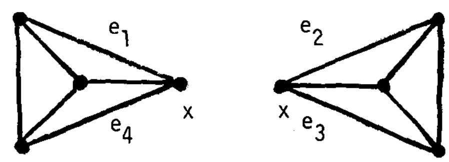


Figure 3.

Note that by the E-P formula the genus has remained unchanged. Later, we may want to reidentify the two copies of x. We could do this by simply constructing the embedding in Figure 2. On the other hand, we could take the left hand copy of K_4 and place it in face F_2 and then reidentify. There are in fact nine ways to identify the two copies of vertex x which produce planar embeddings. We make this more precise in the following definitions and lemmas.

Definition. A graph G' and a morphism $f: G' \to G$ is called a vertex split (or simply a split) of the graph G if f is surjective and injective on P-V.

Thus a vertex split G' of G can be thought of as a graph whose vertices bear the vertices of G as labels. The passage from G' to G is obtained by identifying in G' the vertices with the same label.

A pair of vertice (u,v) is an identification of f if $f(u) = f(v)$.

We may also assume that G' has no isolated vertices.

Note that if G_I is an embedded graph than I can be lifted in a canonical way to an embedding I' of the vertex splitting G' of G. On the other hand an embedding I' of G' may induce a more than one embedding of G.

Definition. A pair (G_I', f) is an embedding of G with splitting if G_I' is an embedded graph and (G', f) is a vertex split of G.

An embedding with splitting corresponds actually to a collection of embeddings of G. For our purposes all these will turn out to be equally good.

There are two natural ways to define the class of embeddings of G induced by (G_I', f). One is global while the other is procedural. For now we shall give only the procedural definition.

Definition. Let G_I be some embedded graph and $\hat{x}$ and $\hat{y}$ two elementary corners of G_I such that $x \neq y$. We define a new embedded graph $G_I/\hat{x}=\hat{y}$ as follows:

The underlying graph, $G/x=y$, is just the usual graph gotten by identifying the vertices x and y. The embedding I' is given by I except at vertex $x=y$ where we now define it. Now I, $\hat{x}$, and $\hat{y}$ determine a linear order of the edges of x and respectively y say these linear orders are $e_1^x, \ldots, e_k^x$ and $e_1^y, \ldots, e_{k'}^y$. The order at $x=y$ will simply be the cyclic order $\langle e_1^x, \ldots, e_k^x, e_1^y, \ldots, e_{k'}^y \rangle$. This operation will be called an identification of corners.

Using this definition we get the following bounds on the genus of $G_I/\hat{x}=\hat{y}$.

Lemma 1. The genus of $G_I/\hat{x}=\hat{y}$ equals

(1) $1 + \text{genus}(G_I)$ if $\hat{x}$ and $\hat{y}$ are corners of different faces and $\hat{x}$ and $\hat{y}$ are in the same component of G.

(2) Genus (G_I) if x and y are corners of the same face or x and y are in different components of G.

Proof. The proof is a simple application of E-P formula.

We shall say that (G_I', f) induces an embedding I on G if G_I can be gotten from $G_{I'}'$ by some sequence of identifications of corners consistent with the identifications of f.

Definition. An embedding with splitting $G_{I'}'$ is an embedding of G of genus g if all induced embeddings of G are of genus at most g.

The important fact about these definitions are summarized in the following lemma.

Lemma 2. Let G and G' be graphs with respectively p and p' components. Let $G_{I'}'$ be an embedding with splitting of G and let G_I be an induced embedding of G. Then genus $(G_I) \leq \text{genus}(G_{I'}') + (v'-v) + (p'-p)$.

As an example of these ideas let $G \simeq K_{3,3}$ with vertices $x_1, x_2, x_3, y_1, y_2, y_3$ and edge common to pairs (x_i, y_j) for $1 \leq i, j \leq 3$.

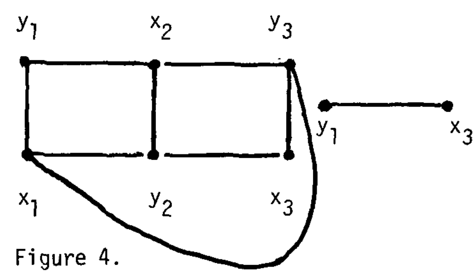


Figure 4.

Now by making two copies of vertices y_1 and x_3 and associating the edges with the vertices as in Figure 4 and also cyclicly ordering the edges as in Figure 4 we get an embedding of $K_{3,3}$ of genus 1. Note that there are two ways to identify corners of y_1 and x_3 respectively. Thus in the sense of Edmonds' this embedding of $K_{3,3}$ represents 4 Edmonds' embeddings of $K_{3,3}$.

An embedded graph G_I is quasiplanar if no vertex appears more than once on any face. Note that by successive vertex splittings any embedding can be transformed into a quasiplanar embedding. A quasiplanar embedding gotten by splitting will sometimes be called a fully split embedding.

From now on, unless we explicitly say otherwise, all embeddings will be embeddings with splitting.

3. HOMOLOGY AND BASIC SUBGRAPHS

3.1. Homology

Let L be a subgraph of G. We shall say L is an equally connected subgraph of G if for each pair of points x,y in L if x and y are in the component of G then x and y are in the same component of L. Given an embedding I of G we can in a natural way view I as an embedding of L, which we will denote by L_I. The embedded graph L_I is said to span G_I if L is a equally connected subgraph of G and $\chi(L_I) = \chi(G_I)$. If L_I is a minimal span of G_I (under the subgraph relation) then L_I is called basic.

In this section our goal is to generate enough homology theory to exhibit basic subgraphs of G_I and find bounds on the number of edges and vertices of reduced basic subgraphs.

The edge space is the vector space consisting of all linear combinations over $GF(2)$ of edges of G [Be 76]. Any cycle or chains can be viewed as an element of the edge space. Let C_1, the cycle space, be the subspace generated by the cycles of G. Then the cycle space is of dimension $e - v + p$. Let T be a spanning forest of G. Now each edge $e \varepsilon G - T$ determines an elementary cycle of G. These cycles form a basis for C_1 which we shall call a spanning forest basis and which will be denoted by S_T. Let G_I be an embedded graph of genus g. Let B_I, called the boundary space, be the subspace of C_1 which is generated by the faces of G_I. The space B_I is of dimension $f - p$ where f is the number of faces. The homology group is the quotient space $C_1/B_I = H_I$ where for us, H_I is simply a vector space over $GF(2)$ of dimension $e - v - f + 2p$.

To obtain Euler's formula we need only the fact that $\dim(H_I)$ is even and set $2g = \dim(H_I)$. This gives $2g = e - v - f + 2p$. If $Z \varepsilon C_1$ let $\overline{Z}$ denote the coset of C_1/B_I containing Z.

Lemma 1. For each embedding I of G of genus g there exist $2g$ cycles $\{Z_1,\ldots,Z_{2g}\}$ in the basis S_T such that $\{\overline{Z}_1,\ldots,\overline{Z}_{2g}\}$ is a basis of H_I.

The proof is a simple induction argument.

Let G be a graph and S_T a cycle basis of G. Let L_k be the union of k cycles, say $Z_1,\ldots,Z_k$, in S_T plus the subspanning forest of T which spans the k cycles in G.

Lemma 2. Let L_I be a subgraph of G_I. If $Z_1,\ldots,Z_k$ represent a set of linearly independent elements of H_I of some graph G_I then $(L_k)_I$ is an embedded graph of genus 2.

This follows from the combinatorial definition of an embedding.

Let v_i be the valence of vertex i in L_k.

Lemma 3. $\sum_{i=1}^{v}(v_i-2) = 2(k-1)$

Proof. The left hand equals $2e - 2v$.

On the right side we know that $\dim(C_1)$ of L_k equals $e - v + 1$ but we have also constructed L_k in such a way that $\dim(C_1) = k$. So $k = e - v + 1$. Substituting this into the right hand side we have

$$2(k-1) = 2(e-v+1-1) = 2e - 2v = \sum_{i=1}^{v}(v_i-2).$$

L_k is a reduced graph with multiple edges and loops. When we count the distinct embedding of L_k we will view L_k as having labelled vertices and edges.

Lemma 4. The number of distinct embeddings of L_k is

$$\prod_{i=1}^{v}(v_i-1)! \leq (2k-1)!$$

Proof. The equality follows by the definition of embedding, while the inequality follows by noting that L_k has a maximum number of embeddings when there is but one vertex of valence $2k$.

Lemma 5. The number of edges of $L_k \leq 3(k-1)$.

This follows by noting that the maximum edges occurs when all vertices have valence ≤ 3. In this case L_k is a cubic graph with $2(k-1)$ vertices and, therefore, it has $3(k-1)$ edges.

3.2. What We Learned From Homology.

We have found that given an embedded graph G_I of genus g and a spanning forest T of G then there exist a basic subgraph L_{2g}. We restate this as an algorithm:

We shall introduce two procedures namely pick and guess. The procedure pick (A) is to arbitrarily or systematically choose some element from A. While guess (A) is to try all possible elements in A.

Procedure: Generate Basic Subgraph (G,g).

(1) Pick a spanning forest T of G.

(2) Guess $2g$ edges from G-T.

(3) Generate L_{2g} from the 2g edges.

(4) Guess an embedding I of L_{2g} of genus g.

Now step (2) contributes a factor of $\binom{e-v}{2g}$ or $O(e^{2g})$ to the running time. While (4) contributes a factor of (4g-1)!. Thus the overall contribution is a factor of $O((4g)!e^{2g})$.

4. EXTENSION PROBLEMS

4.1. Extensions

By the last section we can find a basic subgraph of G_I. In the next two sections we show how to extend the embedding from a basic subgraph to the whole graph.

A partial embedding is a triple (G,L_J,f) where G is a graph and (L_J,f) is an embedding of some equally connected subgraph L' of G. We will often denote this by (G,L,J). An extension of a partial embedding (G,L_J,f) is an embedding (G_I',f) of G such that (a) in a connical way f' is an extension of f and I is an extension of J, and (b) genus $(G_I',f') \leq$ genus (L_J,f). The extension problem is to exhibit an extension if one exists. A quasiplanar extension problem is an extension problem where L_J is quasi-planar.

In this section we show how to "prudently" guess from a partial imbedding (G,L,J) a "partial" extension J' of J to some intermediary subgraph L' such that $(G,L_J',)$ is extendable if and only if (G,L_J) is extendable.

4.2. Regions.

A directed cycle E is the boundary of a region of G_I if there exist a spanning subgraph L of G_I such that E is a face of L_I. So if E is a region then we may talk about the corners and sides of E. For an embedded graph the corners of a given embedding are partially ordered under inclusion and so we may talk about one corner being contained in another. Given two corners $\hat{x}$ and $\hat{y}$ of E they partition the cycle C into two chains $\hat{x}E\hat{y}$ and $\hat{y}E\hat{x}$. We shall use the notation $(\hat{x}E\hat{y})$ and $[\hat{x}E\hat{y}]$ to denote the open respectively the closed chain from $\hat{x}$ to $\hat{y}$. If a and b are two sides of E then [aEb] will denote the closed chain from a to b not including a or b. The interior of E is simply the subgraph generated by all points of G-E embedded "in" E. By making multiple copies of the corners and sides of E we can view E plus its interior as an embedded planar graph E_I. We can partially order regions of G_I under containment. And we obtain the convenient fact that a region E is minimal iff E is a face of G_I. A vertex of edge is said to be internal if it appears more than once on E. We shall say a face F spans two corners $\hat{x}$ and $\hat{y}$ of E if F has two corners one contained in $\hat{x}$ and one contained in $\hat{y}$. A chain $Z \subseteq E_1$ is said to separate $\hat{x}$ and $\hat{y}$ if it is attached to $(\hat{x}E\hat{y})$ and to $(\hat{y}E\hat{x})$.

Lemma 1. If E is a region of G_I with corners $\hat{x}$ and $\hat{y}$ then one and only one of the following conditions are satisfied:

(1) there exist a face which spans $\hat{x}$ and $\hat{y}$.

(2) There exist a chain which separates $\hat{x}$ and $\hat{y}$.

Proof. We first show that (1) and (2) are mutually exclusive. Suppose (1) and (2) are true. Since there exists a face F from $\hat{x}$ to $\hat{y}$ we can add a chain C from $\hat{x}$ to $\hat{y}$ in F without affecting the genus of G_I. We can also discard all other elements from the interior of E except the chain d from condition (2). Now E plus c and d has but one face which contradicts E-P formula.

Suppose that condition (1) is false. Let

$F_1,\ldots,F_k$ be the set of faces generated from corners contained in $\hat{x}$. Now the cycle, in the cycle space C_1, $C = \sum_{i=1}^{k} F_i$ contains the corner $\hat{x}$, is contained in E, and has no corner contained in $\hat{y}$ since none of the F_i's do. So C-F must contain a chain satisfying (2).

Definition. A pair of corners $(\hat{x},\hat{x}')$ of E from some common vertex x is called a cut-point of E if there is a face which spans these two corners. If a is an internal edge of E and x is a point of a, then we say x is a cut-point if there is a face which spans the two corners of x involving a. We now prove the general form of the last lemma.

Theorem 2. If (a,a^R) is an internal pair of region E then one and only one of the following conditions is satisfied:

(1) E has a cut-point on (a,a^R).

(2) There exist two vertex disjoint chains in E from distinct corners of $[aEa^R]$ to distinct corners of $[a^REa]$.

Proof. If x is a cut-point with corners $\hat{x}$ and $\hat{x}'$ then by the previous lemma there exist no chains from $(\hat{x}E\hat{x}')$ to $(\hat{x}'E\hat{x})$ distinct from x so in particular there cannot exist 2 vertex disjoint chains from aEa^R to a^REa.

If there do not exist 2 vertex disjoint chains from $[aEa^R]$ to $[a^REa]$ then there must be a point x which separates these two cycles by Menger's Theorem. Now x must be on a since a connects the cycles. Let $\hat{x}$ and $\hat{x}'$ be the two corners of x common to a. Applying the last lemma to $\hat{x}$ and $\hat{x}'$ either there is a face spanning $\hat{x}$ and $\hat{x}'$ in which case we are done, or there is a chain from $(\hat{x}R\hat{x}')$ to $(\hat{x}'R\hat{x})$. The later implies that in fact x is not a separating vertex.

5. REMOVING INTERNAL EDGES

The last theorem suggest an algorithm for "removing" internal edge e from region E with boundary $<e,x,(a),x,e^R,y,(b),y>$. Let $\hat{x}$ and $\hat{y}$ be the corners of [a] and [b] respectively containing e. We present this in procedure form:

Procedure. Remove Internal Edges (G,L,I).

(1) If L_I has no internal edges output (G,L,I). Pick an internal edge e of some face E of L_I. Let $<ex(a)xe^Ry(b)y>$ be the boundary of E.

(2) Guess 4 edges $(e_1,e_2,e_3,e_4, \varepsilon$ G-L+e) where e_1 is attached to $\hat{x}$
  - e_2 is attached to $\hat{y}$
  - e_3 is attached to a corner of [a] distinct from $\hat{x}$.
  - e_4 is attached to a corner of [b] distinct from $\hat{y}$.

(3) Guess a corner of [a] and [b] in which to embed e_3 and e_4.

(4) Find in G-L+e using an augmenting path algorithm, find one of the following:
  - (a) two vertex disjoint chains from $\{e_1,e_3\}$ to $\{e_2,e_4\}$.
  - (b) Left- and right-most cut points P_1,P_2 from $\{e_1,e_3\}$ to $\{e_2,e_4\}$.

(5) If 4(a) is true then remove e from L and add the two chains to L and embed them (in the unique way) in E. Go to 8.)

Let L' be the subgraph of G-L consisting of all pieces with only attachments in $[P_1eP_2]$ and the chain $[P_1eP_2]$. Let S be the subgraph of L' consisting of those pieces whose attachments are exactly the set $\{P_1,P_2\}$.

(6) If L' is planar then $L \leftarrow cl(L-L')$ else if L'-S is planar then $L \leftarrow cl(L-L+S)$ else return with answer "no".

(7) Replace (xeP_1) and (P_2ey) with the two disjoint chains from (e_1,e_3) and (e_2,e_4) to P_1 and P_2 respectively. Embed these new chains.

(8) Call Remove Internal Edge (G,L,I).

We can analyze the cost of Remove Internal Edge as follows:

Step (2) will add a factor of $O(e^4)$. While Step (3) will add a factor of $O(e^2)$. Now step (4) costs O(e) times the number of guesses so far in the algorithm. But this is bounded by O(e) times all guesses. So the procedure will add at most a factor of $O(e^7)$ steps.

By using this trick as it is we can "remove" all the internal edges from L after one less than the number of internal edges of L.

By adding one more idea we can improve this number of applications by a factor of 2. In this paper we shall only analyze what happens without any new ideas. Let Int(F) be the number of internal edges counted with their multiplicity (actually equal to 2) on some face F. Let $(\mathrm{Int}(L_I) = \Sigma(\mathrm{Int}(F)-1)$ where the sum is over all faces of L_I that contain internal edges.

Lemma 1. Given an extension problem (G,L,I) then after $\mathrm{Int}(L_I)$ recursive calls of Remove Internal Edge to L_I the new embedded graph L_I will be free of internal edges.

Proof. Since Internal edges appear in pairs Int(F) is even and so if $\mathrm{Int}(L_I) = 0$ then L_I has no internal edges. Suppose that $\mathrm{Int}(L_I) > 0$ and e is an internal edge of some face F of L_I. If Remove Internal Edge "cuts" e then after applying the procedure no new internal edge will be introduced and e will no longer be internal. If Remove Internal Edge replaces e with two chains then these chains may partition some internal edge f into two internal edges when they divide E into two regions E' and E". So $\mathrm{Int}(E') + \mathrm{Int}(E") \leq \mathrm{Int}(E)$. But this implies that $\mathrm{Int}(L_I) \geq \mathrm{Int}(L_I') + 1$ where L_I' is L_I after applying the procedure.

6. PARTITIONING INTERNAL VERTICES

In the last section we discuss how to remove internal edges from L. Here we describe a procedure for "removing" internal vertices. A spanning chain Z of a region E is said to separate $\hat{x}_1,\ldots,\hat{x}_K$ if the attachments of Z are distinct from $x_1,\ldots,\hat{x}_K$ and Z separates at least two of the corners of $\hat{x}_1,\ldots,\hat{x}_K$.

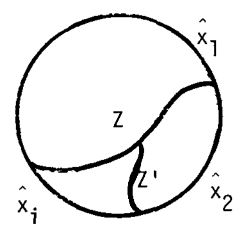
We first generalize Lemma 1 Section 4.

Lemma 1. If F is a region of an embedded graph G_I and $\hat{x}_1,\ldots,\hat{x}_k$, $k \geq 2$ are distinct corners of F then one of the following conditions are satisfied.

(1) There exists a face which spans two corners of $\hat{x}_1,\ldots,\hat{x}_k$.

(2) There exists a chain which separates $\hat{x}_1,\ldots,\hat{x}_k$.

Proof. The proof is by induction on k. The case k = 2 was proved in the previous section. So suppose that the lemma is true for all k'<k. We may assume that the corners $\hat{x}_1,\ldots,\hat{x}_k$ appear in this order on the boundary of F. Now by the previous lemma applied to $\hat{x}_1$ and $\hat{x}_2$ either they are spanned by a face in which case the lemma is proved or else there exists a chain which separates $\hat{x}_1$ and $\hat{x}_2$. So assume that Z is a chain that separates $\hat{x}_1$ and $\hat{x}_2$ and let e_1 and e_2 be the first and final edges of Z. So e_1 is attached to a corner in $(\hat{x}_1F\hat{x}_2)$ and e_2 is attached to a corner in $(\hat{x}_2F\hat{x}_1)$. Now e_1 cannot be embedded in the corners $\hat{x}_3,\ldots,\hat{x}_k$ since $\hat{x}_1$ and $\hat{x}_2$ are "consecutive". If e_2 is also not embedded in the corners then again the lemma is proved. So we must assume that e_2 is embedded in corner $\hat{x}_i$ for some $3 \leq i \leq K$. The chain Z divides the Region F into 2 subregions, one containing x_1 and one containing x_2. Let F_2 be the region containing x_2. Now F_2 contains the corners $x_2,\ldots,x_{i-1}$ plus the new corner defined by e_2 and x_i. So by induction we can apply the lemma to F_2 with these $k = i - 1$ corners. If condition (1) holds again the lemma is proved so we may assume that there exists a separating chain Z'. If Z' attaches only to the boundary of F then Z' satisfies the lemma. So again we may assume that Z' has an attachment on F and one on Z. Using Z' and part of Z we get a chain satisfying condition (2). This proves the lemma.



Suppose that $\hat{x}_1,\ldots,\hat{x}_k$ is a set of distinct corners of some face F of L_I from an extension problem (G,L_I). Our goal is to successively separate F with either chains or cuts until the corners are partitioned.

The following procedure produces a quasiplanar embedding from L_I.

Procedure: Partition (G,L,I).

(1) If L_I is quasiplanar, output (G,L,I).

(2) Pick a face of F with an internal vertex, say x.

(3) Let $\hat{x}_1,\ldots,\hat{x}_k$ be the corners of F common to x.

(4) Guess 4.1 or 4.2 and execute it.

   (4.1) Guess a chain Z which separates $\hat{x}_1,\ldots,\hat{x}_k$ and add Z and its embedding to L_I.

   (4.2) Guess a pair $(\hat{x}_i,\hat{x}_j)$ which is spanned by some face F and cut L at $\hat{x}_i$ and $\hat{x}_j$. Let L be the resulting graph.

(5) Partition (G,L,I).

By our lemmas the procedure Partition is correct. By the next lemma we will achieve an upper bound on the number of recursive calls the procedure makes for a given face. Let $\langle\hat{x}_1,\ldots,\hat{x}_n\rangle$ be the cyclic ordering of the internal corners of F induced by F. Consider the cyclic sequence of vertices $\langle x_1,\ldots,x_n\rangle$. Let c be the number of distinct vertices. For $c \geq 2$ let k be the number of i's such that $x_i \neq x_{i+1}$ modulo n. For c = 0,1 let k = 2. We shall call k the number of alternations of $\langle x_1,\ldots,x_n\rangle$. Using n, c and k we define the following characteristic.

<u>Definition.</u> Let $\underline{\beta}(F) = n - c + \frac{k}{2} - 1$ where n, c and k are as above for the internal corners $\langle x_1,\ldots,x_n\rangle$ of F. Now let $\underline{\beta}(G_I) = \Sigma\beta(F)$ where the sum is over the faces of G_I.

Using β we can now bound the number of recursive calls of Partition:

<u>Lemma 2.</u> The procedure Partition will add at most $\beta(L_1)$ chains or cuts to L_I.

<u>Proof.</u> The proof follows arguments similar to the proceeding lemma. Namely, we first note that $\beta(H_I) = 0$ implies that H_I is quasiplanar. Second, we show that the characteristic is strictly decreasing with each application of Partition. We leave the details to either the diligent reader or the final paper.

7. ANALYSIS OF REDUCTION TO QUASIPLANARITY

In order to obtain a quasiplanar extension problem from (G,L,I) we shall first apply Remove Internal Edges and then we will apply Partition. To analyze the running time it is useful to know what the effect Remove Internal Edges has on $\beta(L_I)$:

<u>Lemma 1.</u> If (G',L',I') is the output of one recursive call of Remove Internal Edge (G,L,I) then $\beta(L'_{I'}) \leq \beta(L_I) + 1$.

<u>Proof.</u> The proof follows arguments similar to the one used in the preceeding lemmas.

We are now in a position to analyze the cost of reducing our problem to a quasiplanar problem.

We analyze this procedure on input (G,L,I) where L is a basic subgraph. By our previous remarks Remove Internal Edge will contribute a factor of at most $(O(e_G^7))^{\mathrm{Int}(L_I)}$. It can be shown that Partition will contribute a factor of $O(e^4)$ for each recursive call. Now Partition will be called at most $\beta(L_I) + \mathrm{Int}(L_I)$. Since L_I is a basic subgraph it has but one face F and therefore $\mathrm{Int}(L_I) = 2e - 1$ where e is the number of edges of L_I. On the other hand $\beta(L_I) = \beta(F) = n - c + \frac{k}{2} - 1$. We can write n, c, and k in terms of e and v as follows.

$$n = 2e, \quad c = v \quad \text{and} \quad k \leq 2e.$$

So $\beta(L_I) \leq 3e - v - 1$. By the E-P formula for L_I we have that $2g = e - v + 1$.

$\beta(L_I) + \mathrm{Int}(L_I) \leq 4e - 3 + 2g$. By lemma 5 Section 3 we know that $e \leq 3(2g - 1)$. Putting this altogether in a theorem:

<u>Theorem 2.</u> With only a contributing of a factor of at most $O(e^{188g})$ to the running time we can transform a basic extension problem to a collection of quasiplanar extension problem (G,L,I) where L has at most 56g edges.

The 56g edges comes from noting that Remove Internal Edge introduces at most 2 new edges, while Partition introduces at most one new edge per recursive call.

8. THE QUASIPLANAR EXTENSION PROBLEM

In this section we give an algorithm which solves the quasiplanar extension problem (G,L,I) in time polynomial in e_G for fixed e_L. Note that the quasiplanar extension problem is NP-complete [Ga ta, Re tal] if viewed only as a function of e_G. Its complexity is open, for fixed g, when viewed only as a function of e_G.

<u>Definition.</u> Two embeddings C_I and $C_{I'}$ of a piece C in E are <u>similar</u> if the distinct

attachment sides and corners of C_I and $C_{I'}$ are the same.

The embeddings of C in L_I are the dissimilar embeddings of C in L_I.

Suppose that (G,L,I) is a quasiplanar extension problem. Since L_I has no internal vertices every piece has at most one embedding in each face. It is not hard to see, using standard linear time planarity algorithms, that in the quasiplanar case we can determine in O(e+g) time the embedding of a piece in L_I.

If some piece C has 3 or more embeddings then C must be embeddable in 3 or more faces. Since points of H are of valence 2 they can appear on at most two faces. By this observation the attachments of C must be only vertices. So we can bound the number of pieces of G-H which can be embedded in more than three ways by bounding the number of components which have attachments consisting only of vertices. There is one degenerate case. This occurs when we have an unbounded number of components from some vertex x to some vertex y. To get around this degenerate case we will define a notation of 2 pieces being similar. This notation is used in [Re ta2].

Definition. Two pieces C and C' of an extension problem (G,L,I) are said to be similar if the distinct attachment corners and sides of C and C' are the same.

Note that, if C and C' have 3 or more attachments to distinct corners of sides then they must be dissimilar. On the other hand if C and C' are attached to at most two distinct vertices then any embedding of C is also simultaneously an "embedding" of C'. So, the number of similar classes is essentially the number of pieces which can be embedded in 3 or more ways in the quasiplanar case.

Lemma 1. The number of dissimilar pieces of (G,L,I) whose attachments are only vertices is at most 6e - 5f where e,f are the number of edges and faces of L_I.

Proof. We count those with 3 or more dissimilar attachments separately from those with only two. In the case of 3 or more dissimilar attachments we shall use the following characteristic. If $L \leq K \leq G$ then $\alpha(K_I) = \Sigma V_L(F) - 2$ where the sum is over faces of K_I and $V_L(F)$ is the number of vertices of L on F. If K is H plus a collection of pieces of G-L and $\alpha(K_I) = 0$ then K must contain all the peices which have 3 or more dissimilar attachments to L. The proof is by simply noting that $\alpha(K_I)$ strictly decreases by adding a piece of G-L to K. Note that $\alpha(L_I) = 2e - 2f$.

Consider the case of pieces with 2 dissimilar attachments. There are those pieces which attach to consecutive corners of a face and those that do not. For those of the first type there must be at most 2e. For those of the second type we use the characteristic $\alpha'(K_I) = \Sigma(V_L(F) - 3)$ and a proof similar to the first case. Note that $\alpha'(L_I) = 2e - 3f$. Thus our bound (2e - 2f) + 2e + (2e - 3f) = 6e - 5f is achieved.

Now each piece can be embedded in at most e_L ways. So we get that the number of ways of embedding the pieces of G-L which can be embedded in 3 or more ways is bounded by $O(e_L^{6e_L})$.

We can sum up what we have shown in this section by saying that the following procedure will add a factor of at most $g^{O(g)}$ steps to the embedding algorithm and reduce the quasiplanar extension problem to an extension problem where the pieces have at most two embeddings:

Procedure. Quasiplanar (G,L,I)

(1) Return if (G,L,I) is a quasiplanar extension problem.
(2) Determine the pieces $C_1,\ldots,C_K$ of G-H which can be embedded in more than three ways.
(3) Guess for each C_i an embeddable face F of L. Embed C_i in F.
(4) Output $(G, L \cup C_1 \ldots \cup C_K, I)$.

9. 2-CNF AND SIMPLE EXTENSIONS

A simple extension problem is a quasiplanar extension problem (G,L,I) where each piece has at most two embeddings in L_I. We prove the following simple fact:

Theorem 1. Simple extension problems are polynomial time reducible to 2-CNF.

Proof. Let (G,L,I) be a simple extension problem and $C_1,\ldots,C_K$ be the pieces of G-L. We shall associate a propositional variable x_i with each piece C_i. The assignment of true to x_i will correspond to one embedding of C_i while false will correspond to the other. One simply needs to notice that the confliction of two pieces on a given face is a disjunction of two literal.

Note that we can find an instantiation of a 2-CNF formula in linear time. The formulas are of size $O(k^2)$ where k is the number of pieces.

We write this as a procedure:

Procedure. 2-CNF (G,L,I)

(1) Determine the embedding of the pieces.

(2) Construct the 2-CNF formula ϕ

(3) If X is an instantiation of ϕ

(4) Use X to extend L_I to G.

10. SUMMARY

Putting all the procedures together we can obtain our genus algorithm.

Procedure. Embedding (G,g)

(1) Generate Basic Subgraph (G,g), say L_I.

(2) Remove Internal Edges (G,L,I).

(3) Partition (G,L,I).

(4) Quasiplanar (G,L,I).

(5) 2-CNF (G,L,I).

We can now analyze the running time of Embedding. We list the multiplicative factors for each of the steps (1) to (4):

(1) $O((eg).e^{2g})$

(2) and (3) $O(e^{188g})$

(4) $O((56g)^{336g})$.

Note that each of these terms is bounded by $(g\cdot v)^{O(g)}$. We state this as a theorem:

Theorem 1. There exists an algorithm to determine the genus of graph which runs in $(g\cdot v)^{O(g)}$ time.

By running Embedding on inputs for successively larger g we can determine the genus of a graph.

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