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The Graph Genus Problem Is NP-Complete

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It is NP-compete to tell, given a graph G and a natural number k, whether G has genus k or less. © 1989 Academic Press, Inc.

1. INTRODUCTION

The *genus* $g(G)$ of a graph G is the smallest number g such that G can be embedded on the orientable surface of genus g. Given a graph G and a natural number k one may ask: Is $g(G) \leq k$? This problem, called the *graph genus problem*, is one of the remaining basic open problems, listed by Garey and Johnson [2], for which there is neither a polynomially bounded algorithm nor a proof that the problem is NP-complete. For k fixed, Filotti *et al.* [1] described a polynomially bounded algorithm for the graph genus problem. Such an algorithm also follows from the Robertson–Seymour theory on minors [5]. The author [6] proved that a given embedding is of minimum genus provided all the noncontractible cycles are longer than all facial walks. [6] also contains both a polynomially bounded algorithm for deciding if a given embedding has this property and also a polynomially bounded algorithm for deciding if a 2-connected graph has an embedding of this type.

However, we shall here prove that the graph genus problem is NP-complete. We show that the problem of deciding if the independence number $\alpha(G)$ (that is, the cardinality of a largest set of pairwise nonadjacent vertices in the graph G) is greater than k (a problem which is known to be NP-complete [2]) can be reduced, in polynomial time, to the graph genus problem. The reduction is as follows: We let G' be obtained from G by replacing each edge xy by a large double wheel. That is, we delete xy and add a long cycle C and all edges between C and $\{x, y\}$. We let G'' be obtained from G' by adding a new vertex and joining it to a vertex in each

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of the new cycles. Then we prove that G'' has genus $q - \alpha(G)$, where q is the number of edges of G. No knowledge of embeddings will be assumed.

2. Basic Properties of Embeddings

We shall treat embeddings purely combinatorially as in [6]. A *graph* has no loops or multiple edges. A *multigraph* may have multiple edges but no loops. A *rotation system* of multigraph with vertex set $\{v_1, v_2, \dots, v_n\}$ is a collection $\Pi = \{\pi_1, \pi_2, \dots, \pi_n\}$ such that π_i is a cyclic permutation of the edges incident with v_i. *An embedded multigraph* is a pair (G, Π) where G is a connected multigraph and Π is a rotation system of G. We shall refer to (G, Π) as the Π-*embedding* of G. π_i is called the Π-*clockwise orientation around* v_i. A π-*facial walk* is a sequence $x_0\ e_0\ x_1\ e_1\ \cdots\ x_{r-1}\ e_{r-1}\ x_r$ (which we abbreviate as $x_0\ x_1\ \cdots\ x_r$) of vertices and edges, where the indices are expressed modulo r and e_i is an edge joining x_i and x_{i+1} for $i = 0, 1, \dots, r-1$. Moreover, if $x_i = v_j$, then $\pi_j(e_{i-1}) = e_i$. The Π-*genus* $g(G, \Pi)$ of the Π-embedding is defined by Euler's formula

$$n - q + f = 2 - 2g(G, \Pi)$$

where n, q and f are the number of vertices, edges and Π-facial walks, respectively. The *genus* $g(G)$ is the minimum of the genera taken over all embeddings of G. This purely combinatorial definition of embedding and genus can easily be shown to be equivalent with the usual topological definition (see, e.g., [3]). If H is a connected subgraph of the connected multigraph G, then an embedding Π of G induces an embedding of H which we shall also refer to as Π. Now H can be obtained from G by successively deleting edges and endvertices such that each multigraph in the sequence is connected. After each operation, the Π-genus is either unchanged or decreased by 1. Hence

(1) $g(G, \Pi)$ is a nonnegative integer and, for any subgraph H of G,

$$g(H, \Pi) \le g(G, \Pi).$$

In particular,

$$0 \le g(H) \le g(G).$$

Similarly, we may consider an embedding Π of G and add a new edge e between two distinct vertices u, v. By "inserting" e in the clockwise orientation around u and v, respectively, we either increase or decrease the number of facial walks by one. If u and v are on the same Π-facial walk

W, then we may add e such that $W \cup \{e\}$ is partitioned into two facial walks in $G \cup \{e\}$. Hence

(2) $g(G) \le g(G \cup \{e\}) \le g(G) + 1$. Furthermore, the first inequality is an equality if G has an embedding of genus $g(G)$ such that u and v are on the same facial walk.

Since $g(T) = 0$ for any spanning tree of G, (2) implies

(3) $g(G) \le q - n + 1$.

In general, (3) is a poor upper bound for $g(G)$. However, we shall describe a general family of graphs for which an inequality analogous to (3) becomes an equality.

If W is a (closed) Π-facial walk, then we define the length m of W as the number of edges in W, where an edge is counted twice if it is traversed in both directions in W. We define the *excess* of W to be $m - 3$. The *Π-facial excess* $\mathrm{fe}(\Pi, G)$ of G is the sum $\Sigma(m - 3)$ taken over all Π-facial walks. Since $2q = \Sigma m$, Euler's formula implies

(4) $\mathrm{fe}(\Pi, G) = 6g(G, \Pi) - 6 + 3n - q$.

We say that a cycle C in a connected graph G is *induced* if it has no chords (i.e., edges joining nonconsecutive vertices on C) and *nonseparating* if $G - V(C)$ (i.e., the graph obtained from G by deleting all vertices of C and their incident edges) is connected. Now (4) implies the following crucial result:

(5) If the connected graph G has an embedding of facial excess p, and C is an induced nonseparating cycle in G of length at least $p + 4$, then

$$g(G - V(C)) < g(G).$$

Proof of (5). Let Π be an embedding of G of genus $g(G)$. By (4),

$$\mathrm{fe}(\Pi, G) \le p.$$

Let the notation be such that $C: v_1 v_2 \dots v_r v_1$, where $r \ge p + 4$. We say that an edge incident with v_i is on the *left side* of C if it is one of the edges $\pi_i(e), \pi_i^2(e), \dots, \pi_i^{t-1}(e)$, where e is the edge $v_i v_{i-1}$ and $\pi_i^t(e) = v_i v_{i+1}$. An edge incident with v_i which is not on C and not on the left side is said to be on the *right side* of C. Since $\mathrm{fe}(\Pi, G) \le p$ and C has length $\ge p + 4$, C is not contained in a Π-facial walk. Therefore, there are edges on the left side and edges on the right side of C. We "cut" C into two cycles as follows: We add to G a cycle $C': v_1' v_2' \cdots v_r' v_1'$. If $e = v_i u$ is an edge on the right side of C we delete e and add instead the edge $e' = v_i' u$ which we refer to as the "same edge" as e. (Since C is induced, u is not on C.) The resulting graph is denoted by G'. We modify Π into an embedding Π' of G' in the obvious way: Π and Π' agree for vertices in G except that in Π', the successor of $v_i v_{i+1}$ around v_i is $v_i v_{i-1}$. The Π'-orientation around v_i' is the same as the Π-orientation around v except that the successor of $v_i' v_{i-1}'$ is $v_i' v_{i+1}'$. Since C is induced and nonseparating, G' is connected. Every

Π-facial walk in G is a Π'-facial walk in G' (except that some occurrences of v_i may be replaced by v_i'). In addition, both C' and (the reverse of) C are Π'-facial walks in G'. Hence

$$g(G', \Pi') = g(G, \Pi) - 1.$$

By (1),

$$g(G - V(C)) = g(G' - (V(C) \cup V(C'))) \leq g(G')$$

and hence

$$g(G - V(C)) \leq g(G', \Pi) = g(G, \Pi) - 1 < g(G). \square$$

3. Reducing the Vertex Independence Problem to the Genus Problem

The inequality of (5) enables us to construct a large class of graphs for which we can calculate the genus. If u and v are distinct vertices in a connected graph G, then we may form the disjoint union of G and a cycle C of length of m and add all edges between $\{u, v\}$ and C. We say that the resulting graph G' is obtained from G by *adding a double wheel of order m* between u and v. If the edge $e = uv$ is present, then we say that $G' - e$ is obtained from G by *replacing e by a double wheel of order m*.

(6) If G is a connected graph with n vertices and q edges and G' is obtained by adding a double wheel of order $m \geq 5q - 3n + 10$ between two distinct vertices u and v in G, then

$$g(G') = g(G) + 1.$$

Proof of (6). Let Π be an embedding of G of genus $g(G)$. By (3) and (4),

$$\text{fe}(\Pi, G) \leq 5q - 3n.$$

Let e_1, e_2 (resp. e_3, e_4) be two consecutive edges in the Π-clockwise ordering around u (resp. v). Let $C: u_1u_2 \dots u_m u_1$ be the cycle in $G' - V(G)$. We modify the embedding Π of G into an embedding Π' of G' as follows: Π' agrees with Π except that $e_1, uu_1, uu_2, \dots, uu_m, e_2$ and $e_3, vu_m, vu_{m-1}, \dots, vu_1, e_4$ are sequences in the Π'-clockwise orientation around u and v, respectively. In addition, the Π'-clockwise orientation around u_i is u_iu, u_iu_{i-1}, u_iv, u_iu_{i+1} for $i = 1, 2, \dots, m$ (where the indices are expressed modulo m). All Π-facial walks of G are Π'-facial walks except those two which contain e_1, e_2 and e_3, e_4, respectively. These two Π-facial walks (which may be the same) are enlarged by the edges uu_1, u_1u_m, u_mu and vu_m, u_mu_1, u_1v, respectively. All edges on or incident with C (except $u_1\ u_m$) are in Π'-facial walks of length 3.

Let n, q, f be the number of vertices, edges, and Π-facial walks, respectively, of G. Let n', q', f' be the corresponding numbers for G' and Π'. Then

$$n' = n + m, \qquad q' = q + 3m, \qquad f' = f + 2m - 2.$$

Hence

$$g(G', \Pi') = g(G, \Pi) + 1 = g(G) + 1$$

and

$$g(G) \leq g(G') \leq g(G) + 1.$$

Since

$$\mathrm{fe}(\Pi', G') = \mathrm{fe}(\Pi, G) + 6 \leq 5q - 3n + 6$$

it follows from (5) that

$$g(G) = g(G' - V(C)) < g(G').$$

Hence $g(G') = g(G) + 1$. □

We now prove a result with some analogy to (3).

(7) Let G be a connected graph with n vertices and q edges and let F be a forest (i.e., a graph with no cycle) in G. Let G' be the graph obtained from G by replacing each edge of G outside F by a double wheel of order m, where $m \geq 3n^2$. Then

$$g(G') = q - n + 1.$$

Proof. Let T be a spanning tree containing F and let T' be obtained from T by replacing each edge of T outside F by a double wheel of order m. It is easy to see that T' has a planar (i.e., genus zero) embedding such that the facial excess is at most $3n - 6$. (T has an embedding of genus zero with facial excess $2(n - 1) - 3$ and whenever an edge of T is replaced by a double wheel the facial excess is increased by one.) Now G' is obtained from T' (which is connected) by successively adding double wheels. Since the facial excess of T is at most $3n - 6$ and the facial excess is increased by 6 whenever we add a double wheel as in the proof of (6), and since we add to T' at most $\binom{n}{2} - (n-1)$ double wheels we conclude that the current graph has facial excess at most $3n - 6 + 6\binom{n}{2} - 6(n-1) = 3n^2 - 6n$. Since $m \geq 3n^2$, we conclude as in the proof of (6) that the genus is increased by 1 whenever we add a double wheel of order m to T'. Hence

$$g(G') = q - n + 1. \square$$

We are now ready for the construction which relates the genus $g(G)$ to the independence number $\alpha(G)$, i.e., the maximum cardinality of a set of pairwise nonadjacent vertices in G.

(8) Let G be any connected graph with n vertices and q edges. Let G' be obtained by replacing every edge of G by a double wheel of order $3(n+1)^2$. Let G'' be obtained from G' by adding a new vertex u and joining u to one vertex of each cycle in $G' - V(G)$. Then

$$g(G'') = q - \alpha(G).$$

Proof of (8). First we shall describe an embedding of G'' of genus at most $q - \alpha(G)$. We apply (7) (with $F = \emptyset$) in order to find an embedding Π of G' of genus $q - n + 1$. Consider now a Π-facial walk W. Either W is a triangle in a double wheel or else W is composed of $d(v)$ triangles belonging to $d(v)$ distinct double wheels containing the vertex v in G, where $d(v)$ is the degree in G of v. (This follows from a close inspection of the proof of (6), (7). First we verifiy, for example, by induction on the number of vertices of T', that T' has a planar embedding such that each facial walk containing a vertex v of G is either a triangle in a double wheel or consists of a collection of traingles, one from each double wheel in T' containing v. Then we show that this facial structure can be preserved when we successively add double wheels to T' in order to get G'.) In the latter case W has the length $3d(v)$. Thus the facial excess of Π is $\Sigma_{v \in V(G)}(3d(v) - 3) = 6q - 3n < 3n^2$. Now let S be a set of $\alpha(G)$ independent vertices of G. For each vertex v in $V(G) \setminus S$ we consider the Π-facial walk W_v consisting of $d(v)$ traingles containing v. We add to G' a new vertex v' and join it to one vertex (distinct from v) in each of these $d(v)$ triangles. Since the neighbours of v' are on the same Π-facial walk, an easy extension of the remark preceding (2) shows that Π can be modified to an embedding of the same genus of the graph obtained from G' by adding v' and its incident edges. We do this for every vertex v in $V(G) \setminus S$. By (2), adding an edge increases the genus by at most one. Clearly, the contraction of that edge does not increase the genus further. Hence identifying two vertices increases the genus by at most one. So, if we identify all the new vertices v' added to

G' into one vertex u, then the resulting graph H has an embedding of genus at most $g(G', \Pi) + n - \alpha(G) - 1 \leq q - \alpha(G)$.

Since every edge of G is incident with a vertex of $V(G) \setminus S$, it follows that H contains G'' as a subgraph. Hence

$$g(G'') \leq g(H) \leq q - \alpha(G).$$

Now let Π' be an embedding of G'' of genus $g(G'')$. By the above upper bound on $g(G')$ and (4) we have

$$\mathrm{fe}(\Pi', G'') \leq 5q - 12 + 3n < 3n^2 - 12.$$

Hence no cycle C in $G' - V(G)$ is contained in a Π'-facial walk. Consider the Π'-facial walk W containing the edge from u to C. Since W does not contain C, W contain s a vertex $v(C)$ which is not in C but in the double wheel containing C. The remark preceding (2) shows that Π' can be modified to an embedding of $G \cup \{uv(C)\}$ of genus $g(G'')$. We add the edge $uv(C)$ for every cycle C in $G' - V(G)$. The resulting graph H' has genus $g(G'')$, and H' contains the graph H'' obtained from G' by adding the vertex u and joining it to all vertices of the form $v(C)$. Let Y denote the set of vertices of the form $v(C)$ and put $y = |Y|$. Since each edge of G is incident with a vertex of Y, the set $V(G) \setminus Y$ is independent in G and has therefore cardinality $\leq \alpha(G)$. Hence $y = |Y| \geq n - \alpha(G)$. Note that H'' can be obtained by first joining u to y vertices of G and then replacing the edges of G by double wheels of order $3(n+1)^2$. We now apply (7) (with F consisting of the edges incident with u) to conclude that

$$g(H'') = (q + y) - (n + 1) + 1 \geq q - \alpha(G).$$

Hence

$$g(G'') = g(H') \geq g(H'') \geq q - \alpha(G)$$

and the proof is complete. □

Since the graph G'' is obtained in polynomial time from G and since it is NP-complete to decide if $\alpha(G) \geq k$ (where G is any graph and k is any natural number), we conclude

THEOREM. *The following problem is NP-complete: Given a graph G and a natural number k. Decide if $g(G) \leq k$.*

If G is a graph and k is a natural number, then we may form a new graph H by adding a set S of k new independent vertices each of which is joined to all vertices of G. Now the statements (i) and (ii) below are equivalent:

(i) $\alpha(G) > k$;

(ii) S is not a maximum independent set in H.

The proof of (8) shows that is we know a set S of k independent vertices in G, then we obtain an embedding of G'' of genus $q - k$. Hence it is NP-complete to decide if a given embedding is of minimum genus. A natural approach to this problem is to modify the clockwise orientation locally. Gross and Tucker [3] observed that there exist nonminimum-genus embeddings which cannot be modified to embeddings of smaller genus just by modifying the clockwise orientation around one (well-chosen) vertex. We point out that it is not even sufficient to modify the clockwise orientation around 10^{10} well-chosen vertices. Consider the type of graph G indicated in Fig. 1 embedded on the torus which we think of as rectangle whose opposite sides are identified. This graph G is planar. But in every planar embedding of G all the facial walks are cycles of length 3 or $3n/4$, where $n = |V(G)|$. So, in order to obtain a planar embedding of G it is necessary to modify the embedding of Fig. 1 around some vertex of every 4-cycle. That is, at least $n/4$ vertices will be affected. Gross (private communication) has informed the author that an iteration of the construction in Example 3.5.1 in [3] also results in a graph that requires arbitrarily many rotation changes to reach the global minimum.

Miller [4] asked if a polynomially bounded algorithm for determining the genus of a 3-connected graph implies a polynomially time algorithm for all graphs. If G is a 2-connected graph, then the graph G'' in (8) is 3-connected. Since the problem: "Is $\alpha(G) \leq k$?" is NP-complete for 2-connected graphs it follows that the graph genus problem is NP-complete also when restricted to 3-connected graphs. This answers Miller's question in the

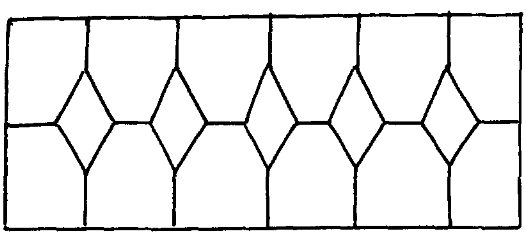

FIGURE 1.

affirmative. Also, our proof gives the stronger result that it is even NP-complete to decide if a given embedding of a 3-connected graph is of minimum genus.

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