5 Recursion

What's in This Set of Notes?

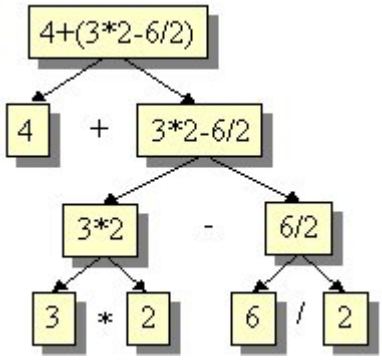
We will now take a short break from GUI design and look at an important programming style/technique known as **Recursion**. In nature (and in mathematics), many problems are more easily represented (or described) using the notion of recursion. It is often the case that we program recursively so as to provide a simpler and more understandable solution to the problem being implemented. In fact, many data structures used in computer science are inherently recursive, making recursive programming natural and often efficient.

Here are the individual topics found in this set of notes (click on one to go there):

5.1 What is Recursion ?

One of the most common techniques used in computer science is known as "divide and conquer". This technique represents a strategy of dividing a complex problem into smaller, easier-to-solve sub problems. There are many real-world examples of how we break problems into smaller ones to solve them:

1. Jigsaw puzzles are solved in "steps": border, interesting portion, grass, sky, etc..
2. Math problems are broken down into smaller/simpler problems
3. Even climbing stairs eventually breaks down to climbing one step at a time.



So then what does this have to do with recursion ? Well, recursion applies the divide and conquer strategy. The word ***Recursion*** actually comes from a Latin word meaning "a running back". This makes sense because recursion is the process of actually "going off" and breaking down a problem into small pieces and then bringing the solutions to those smaller pieces back together to form the complete solution. Here are some points to remember:

- recursion **breaks down** a complex problem into **smaller sub-problems**
- subproblems are **smaller** instances of the **same type** of problem.

It sounds a little bit abstract doesn't it ? Why would we want to do this anyway ?

- some problems are **naturally** recursive
  - e.g., especially math problems/functions such as **factorial**
- simpler, **more elegant solutions** are often obtained
- **easier to understand** completed solutions
- can be **only way to approach** a seemingly **overwhelming problem**

Do we really need recursion ? Well, any problem that is solved recursively can also be done without recursion, but usually the solution is more complex and it is difficult to consider all special cases.

So, recursion is all about:

1. figuring out how to break the problem down into smaller sub-problems
2. handling the smaller sub-problems
3. figuring out how to merge the results of the smaller sub-problems to answer the original problem

In fact, its actually easier than this sounds. Most of the time, we simply take the original problem

and break/bite off a small piece that we can work with. We simply keep biting off the small pieces of the problem, solve them, and then merge the results.

It is important to remember the following very important facts about the sub-problems:

- must be an instance of the **same kind** of problem
- must be **smaller** than the original problem

So how many **times** do we bite off the small pieces ? When do we know when to stop ?

Its simple. We stop when there are no more pieces ... or when the remaining piece is so simple, that it is easily solved without needing to break it down any further. At this "lowest level", we call this simplest problem the "***base case***" or "***basis case***".

In fact, when writing our code, we will usually start with the base cases since they are the ones that we know how to handle. For example, if we think of our "real world" examples mentioned earlier, here are the base cases:

1. For the jigsaw puzzle, we divide up the pieces until we have just a few (maybe 5 to 10) pieces that form an interesting part of our picture. We stop dividing at that time and simply solve (by putting together) the simple base case problem of this small sub picture. So the base case is the problem in which there are only a few pieces all matching together.

2. For the math problem, we simply keep breaking down the problem until either:

   a) there is a simple expression remaining (e.g., 2 + 3) or
   b) we have a single number with no operations (e.g., 7).

   These are simple base cases in which there is either one operation to do or none.

3. For the stair climbing problem, our base case is our simplest

case ... when there is only one stair. We simply climb that stair.

Tips for Designing Recursive Methods:

1. Decide on a name for the method and what its ***parameters*** should be; you have to do this for non-recursive methods just as well.

2. You must ***believe that the method will work*** before you even begin to implement it; this is important. Without it, you will not have the faith to use the method to solve a simple problem especially if the method isn't finished.

3. You must decide on the ***simple cases*** that can be implemented trivially (the basis cases) and then write the code.

4. Determine a technique for breaking up the more complicated case into simpler parts, some of which can be done by using the original method with simpler parameters. This is usually the hard part. ***Don't think about the recursion***. Just think about how to express the original problem in terms of the smaller one's solution.

It would be a good idea to read the above tips again after you have tried writing a couple of recursive methods.

5.2 Recursion With Primitives

We now need to start looking at some examples of using recursion. We will first consider examples in which the recursion occurs without needing to manipulate any objects. That is, we will look at a couple of examples in which recursion is used to compute some values. The simplest example is that of using the factorial function. In fact, the factorial example is the

"hello world" example of recursion since the factorial function is a very natural and simple operation that is inherently recursive. By now, most of you know what the factorial operation does:

5! = 5*4*3*2*1
4! = 4*3*2*1
3! = 3*2*1
2! = 2*1
1! = 1
0! = 1

The operation is defined non-recursively as follows:

$$N! = \begin{cases} 1 & \text{if } N = 0 \\ N \times (N-1) \times (N-2) \times \ldots 3 \times 2 \times 1 & \text{if } N >= 1 \end{cases}$$

We can easily write a **factorial()** method that takes an integer and computes the factorial using loops.

```
// A non-recursive method for computing the factorial
public static int factorial(int num) {
    int  result = 1;
    for (int i=2; i<=num; i++)
        result *= i;
    return result;
}
```

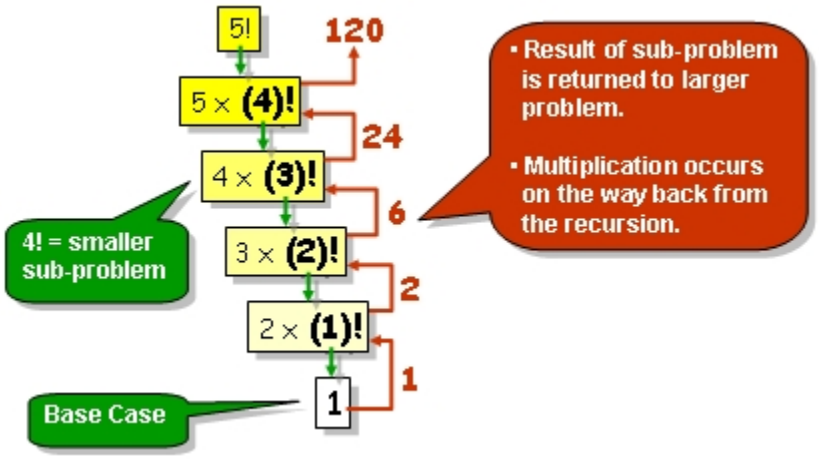
How could we write this code recursively ? First, we must figure out how to express the problem recursively. In this case it is easy since there is a well-known recursive definition:

$$N! = \begin{cases} 1 & \text{if } N = 0 \\ N \times (N-1)! & \text{if } N >= 1 \end{cases}$$

Notice here that **N!** is defined in terms of a smaller factorial problem ... that of **(N-1)!**. So we in fact reduce our initial number N by 1. Then how do we solve the (N-1)! problem ? Well, just apply the same formula:

$$(N-1)! = (N-1) \times ((N-1)-1)! = (N-1) \times (N-2)!$$

So then we need to break down (N-2)! ... which is done in the same way. Eventually, as we keep reducing **N** by 1 each time, we end up with **N=0 or 1** and for that simple problem we know the answer is 1. So breaking it all down we see the solution of **5!** as follows:



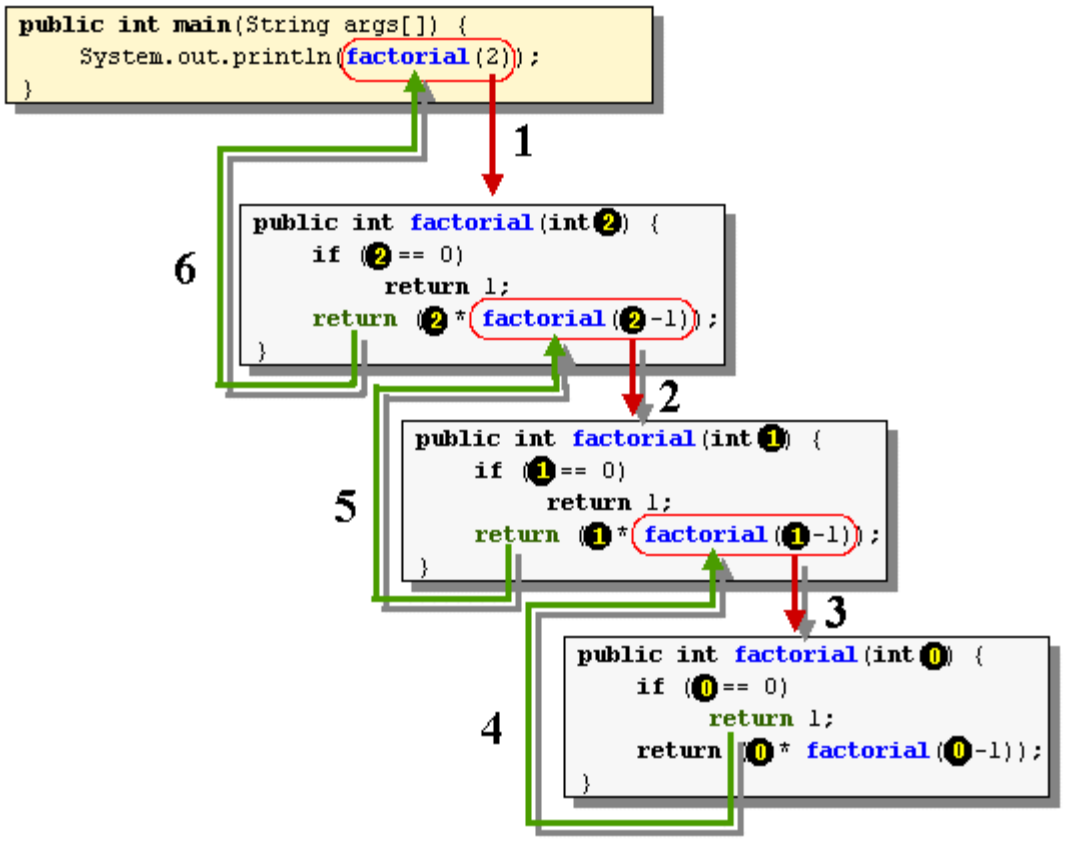
So how do we write the code ? Its easy. You start with the base case. Then use the formula to break down the problem:

```
public static int factorial(int n) {
    if (n == 0)  // BASE CASE
        return 1;
    return n * factorial(n - 1);  // RECURSIVE STEP
}
```

Wow! That's a simpler solution than the non-recursive version. Did you notice how the method actually calls itself ?

Recursive methods actually work just like your "normal everyday" methods. There is nothing tricky. In essence, each time a message is sent, a new "copy" of the method runs on a new copy of the parameters. The local variables of one method invocation are not visible to another invocation.

Consider tracing the recursive execution of "2 factorial":

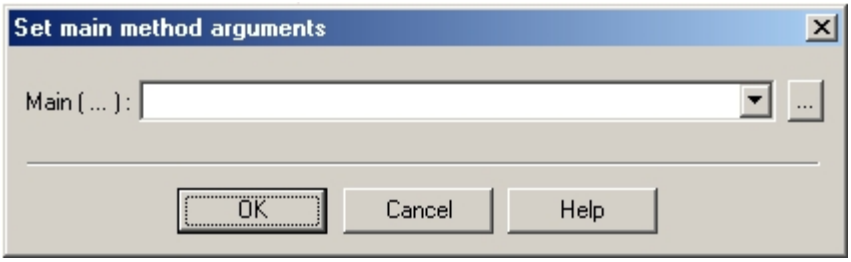


Below is the code as a test application. In it, we will make use of something new called ***command-line arguments***.

Command-line arguments:

- are specified when running a JAVA program
- provide some additional information similar to the notion of parameters.
- are passed as String parameters to the **main** method, stored in **`String args[]`**.
- we can ask **`args`** for its length to see if there are any command line arguments.
- you may not use spaces in your String parameters since a space character is used to separate the parameters.

In JCreator, we can set the command line arguments by going to the **Configure**/**Options** menu and selecting **JDK Tools**. Than **select the tool type** to be **Run Application**. Select **<Default>**, then press **Edit**. Click on the **Parameters** tab and then select the option "**Prompt for main method arguments**". Press OK twice and then run your application. Now when you run, you will be asked to enter some text for your application:



We will run our code the first time with the value of 5 as the command line argument. So type 5 into the text field. You should get a result of 120. Here is the code:

```
public class FactorialTest {

    public static int factorial(int n) {
        if (n == 0)                     // BASE CASE
            return 1;
        return n * factorial(n - 1);    // RECURSIVE STEP
    }

    public static void main(String[] args) {
        //First check to see that there is at least one command line argument
        if (args.length == 0) {
            System.out.println("Usage: FactorialTest <anInteger>");
            System.exit(-1);
        }
        int theInteger = Integer.parseInt(args[0]);

        if (theInteger < 0) {
            System.out.println("Factorial of a negative int is not defined");
            System.exit(-1);
        }
        else
            System.out.println("Factorial of " + theInteger + " is " +
factorial(theInteger));
    }
}
```

Did you notice how we can check the length of the **args** array to ensure that there is at least one argument ? We then accessed that argument by accessing the String array **args** at its first position (which is the first parameter, which was the number 5 we entered). Be aware though that the value comes in as a String, so we have to parse it into the desired type.

What important test cases are not covered by the code above ? Try entering 16, 17, 100. Are there any problems ? Do you know why ?

Example (Mortgage Payment Calculator):

Consider a second recursion example that computes how much money per month a person would have to pay to pay back a mortgage on his/her home. Consider the following notation:

- **p** = the principal cost/amount borrowed (e.g., $130,000)
- **i** = the annual interest rate as a percentage (e.g., 3.5%)
- **t** = the term (in months) that we wish the mortgage to be for (e.g., 300 months for a 25 year mortgage)
- **m(p, i, t)** = the monthly payments we need to make based on the parameters just mentioned (e.g., $647.57)

We would like to determine the value for **m**. Let us assume that **t > 0**, otherwise the point of calculating a payment is silly anyway.
Certainly if the **t = 1** then we must pay one month interest and so the value of **m** should be **p*(1+i)** ... that is ... one month of interest. Otherwise, we will consider the following recursive formula to calculate the amount to be paid each month:

$$m(p, i, t) = \frac{p}{\left(\frac{p}{m(p, i, t-1)} + \frac{1}{(1+i)^t}\right)}$$

Now there are more accurate and efficient ways to compute mortgage payments, but we will use the above formula so that we can practice our recursion.

So how do we write the code to do this ? Certainly the function we need to write requires 3 parameters:

```
public static double calculatePayment(int p, double i, int
t) {
  ...
}
```

Now let us determine the base case. It actually follows right from the definition.

```
if (t == 1)
    return p*(1+i);
```

That wasn't so bad. Now what about the recursive part ? It also follows from the formula:

```
return p / ((p / calculatePayment(p,i,t-1)) +
(1/Math.pow(1+i, t)));
```

So here is the whole thing:

```
public static double calculatePayment(int p, double i, int
t) {
    if (t == 1)
        return p*(1+i);

    return p / ((p / calculatePayment(p,i,t-1)) +
(1/Math.pow(1+i, t)));
}
```

Does it work ? We should write a test method. Here is a class with our method, along with a main method that reads the command line arguments:

```
public class MortgagePaymentCalculator {
    public static double calculatePayment(int p, double i,
int t) {
        if (t == 1)
            return p*(1+i);

        return p / ((p / calculatePayment(p,i,t-1)) +
(1/Math.pow(1+i, t)));
    }

    public static void main(String[] args) {
        //First check to see that there is at least three
command line arguments
        if (args.length < 3) {
            System.out.println("Usage:
MortgagePaymentCalculator <principal> <interest rate>
<term>");
            System.exit(-1);
        }
        int principal = Integer.parseInt(args[0]);
        double intRate = Double.parseDouble(args[1]);
        int term = Integer.parseInt(args[2]);

        System.out.printf("The monthly mortgage payment for
a %d month mortgage of $%,d " +
                          "home at %1.3f percent annual
interest is $%4.2f per month.",
                          term, principal, intRate,
calculatePayment(principal, intRate/100.0/12.0, term));
    }
}
```

Notice that the user enters the interest rate as an annual rate (e.g., 3.5) and the term is in months. We then need to adjust the interest rate to be a percentage (hence divide by 100) and also to make it monthly to match the term units (hence divide by 12).

So from our two examples, you can see that recursion is quite simple once you have a recursive formula.

5.3 Recursion With Objects (Non-Destructive)

Now that we have seen some simple examples of recursion that dealt only with some primitive calculations, we need to look at how recursion works when objects are involved. Let us look at a simple example of reversing a string.

Example (Reversing a String):

As you may know, strings in JAVA are not mutatable (that is, you cannot actually modify a string once it is made). We will look at a method that takes a string and then returns a new string which has the same characters as the original, but in reverse order:

```
public static String reverse(String s) {
    ...
}
```

We start by considering the base case. What strings are the easiest ones to solve this problem for ? Well, a string with 1 or 0 characters is easy, since there is no reversing to do. So there you have it. Those are the base cases:

```
if (s.length() == 1)
    return s;

if (s.length() == 0)
    return s;
```

We can simplify this to one line:

```
if (s.length() <= 1) return s;
```

Now, how do we express the problem recursively ? Remember ... think of a smaller problem of the same type. A smaller problem would be a smaller string. So what if we take a piece of the string and then solve for the smaller string ? That is the way we should approach it. What piece should we take off ? Perhaps the first character. We can use:

```
s.substring(1, s.length())
```

to get the smaller string which is the original without the first character.

OK. Now what do we do ? Remember, we need to express the solution to original problem in terms of the smaller problem. So we should be thinking the following question: "If I have the reverse of the smaller string, how can I use that to determine the reverse of the whole string ?". Lets look at an example:

"STRING" reversed is "GNIRTS". If we use consider the shorter string "TRING" and reverse it to "GNIRT", then how can we use this reversed shorter string "GNIRT" to obtain the solution "GNIRTS" to the original problem ? You can see that all we have to do is append the "S" to the smaller string solution "GNIRT" to get the complete solution "GNIRTS". So,

```
reverse("STRING") = reverse("TRING") + "S"
```

Now we should know how to write the code:

```
public static String reverse(String s) {
    if (s.length() <= 1) return s;

    return reverse(s.substring(1, s.length())) + s.charAt(0);
}
```

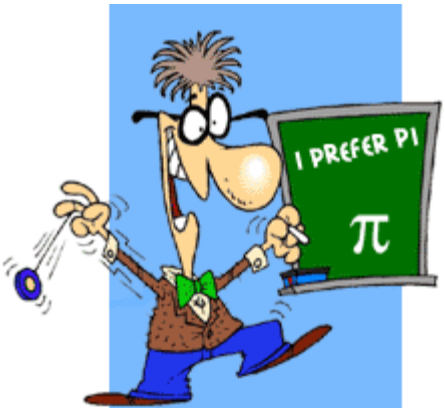
As you can see the solution is quite simple ... **after** we see the solution of course ;).

Here is the test code:

```
public class ReverseTest {
    public static String reverse(String s) {
        if (s.length() <= 1) return s;
        return reverse(s.substring(1, s.length())) + s.charAt(0);
    }

    public static void main(String[] args) {
        //First check to see that there is at least one command line
argument
        if (args.length == 0) {
            System.out.println("Usage: ReverseTest <aString>");
            System.exit(-1);
        }
        System.out.println(args[0] + " reversed is " +
reverse(args[0]));
    }
}
```

Example (Palindromes):

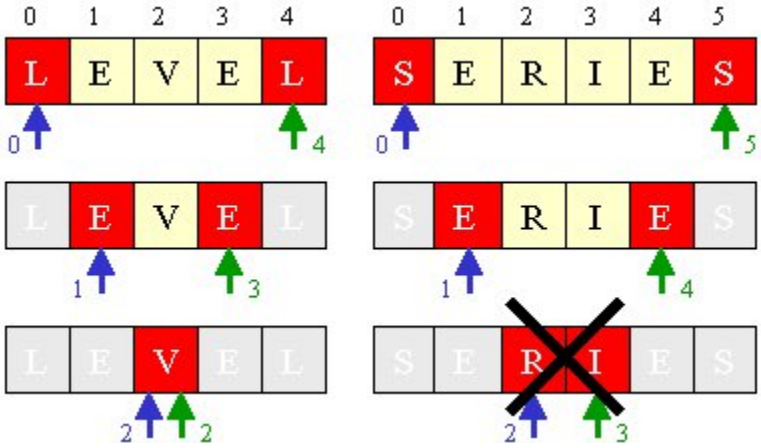
Consider the simple problem of determining whether or not a String is a palindrome. A ***palindrome*** is a String which reads the same forwards as backwards:



- level
- noon
- mom
- madam

How do we write a method using a **for** loop to detect whether or not a String is a palindrome ? Well ... how do we do it without a computer ?

We probably compare the first and last characters and then work our way inward toward the center of the String:



Here is how we may write code to do this:

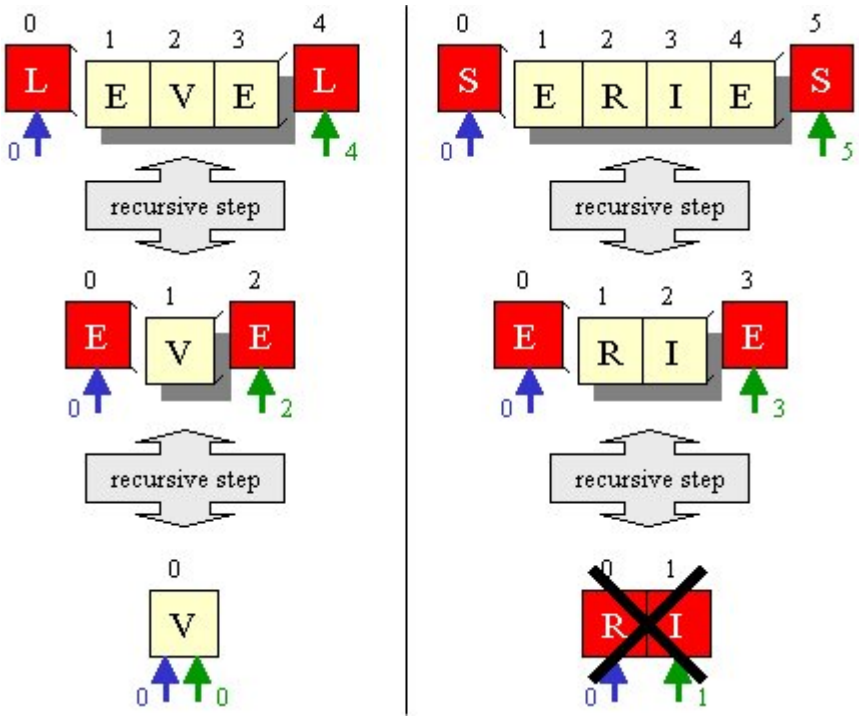
```
public static boolean isPalindrome(String s) {
    for (int i=0; i<=(s.length()-1)/2; i++) {
        if (s.charAt(i) != s.charAt(s.length() - i - 1))
            return false;
    }
    return true;
}
```

Now what about solving this recursively ?

We must be able to express the palindrome problem in terms of smaller palindrome problems. Here is the recursive formulation of the problem:

- A String is a palindrome if its first and last characters are identical AND the substring in between is a palindrome.

Here is how the recursion can be done:



For example, in our palindrome problem, when only 1 character remains in the String, or in the case where the String is empty, we don't need to divide any further and so these are the base cases for our problem. Often, the base cases also correspond similarly with error-checking (e.g., like checking for an empty string first). So ... the ***base case*** is used to "**stop**" the recursive process.

Here are some palindrome examples that show when the base case is reached.

```
isPalindrome("level")  ---> isPalindrome("eve")  --->  isPalindrome("v")  ---
> true
isPalindrome("poop")  ---> isPalindrome("oo")  --->  isPalindrome("")  --->
true
isPalindrome("abcdba")  ---> isPalindrome("bcdb")  --->  isPalindrome("cd")
---> false
```

So how do we write the code for this problem recursively ? Think of the stopping conditions and write pseudo code:

- if the string is empty return true (i.e., an empty string is considered a palindrome)
- if the string has only 1 character in it, return true
- otherwise ... if the first and last characters do not match, return false
- otherwise return the result of the recursive sub-problem on a substring which excludes the first and last characters

Start the code with the base cases, then do the recursive part:

```
public class StringUtilities {
```

```
    public static boolean isPalindrome(String s) {
        //BASE CASE (1 or 0 character cases combined together here)
        if (s.length() <= 1)
            return true;

        //BASE CASE (first and last characters do not match)
        if (s.charAt(0) != s.charAt(s.length() - 1) )
          return false;

        //RECURSIVE STEP (check if middle portion is a palindrome)
        return isPalindrome(s.substring(1, (s.length() - 1)));
    }
}
```

Here is another way to write it:

```
public static boolean isPalindrome(String s) {
    return ((s.length() <= 1) ||
              ((s.charAt(0) == s.charAt(s.length() - 1)) &&
             (isPalindrome(s.substring(1, (s.length() - 1))))));
}
```

Now we must write a test application. Here is some testing code written in a different class:

```
public class PalindromeTest {
    public static void main(String[] args) {
        //First check to see that there is at least one command line argument
        if (args.length == 0) {
            System.out.println("Usage: PalindromeTest <aString>");
            System.exit(-1);
        }
        String input = args[0]; // Access the first argument from the
argument list

        if (StringUtilities.isPalindrome(input))
            System.out.println(input + " is a palindrome ");
        else
            System.out.println(input + " is NOT a palindrome " );
    }
}
```

5.4 Recursion With Objects (Destructive)

In the previous section, we showed how we could break pieces off of Strings each time we called the method recursively. In fact, we did not really alter the string, since the **substring()** method actually returns a new string. As a result, we did not really destroy the original string. Sometimes, however we might want to use a destructive modification of an object. For example we might want the actual elements of an object to be altered, instead of making a new object that represents the changes to the original.

- ***Destructive methods*** - alter the receiver or parameters in some way to obtain result.
- ***Non-destructive methods*** - usually creates new objects to contain the "answer" to the problem, leaving the receiver and/or parameters intact.

Often, it is easier to destroy an object when performing an operation. For example, if someone asked you to count the jelly beans in a big jar you would probably not leave the jar intact. Instead, you'd alter it by removing the jelly beans one at a time (or in small amounts) and do the counting while placing the counted jelly beans in a new and initially empty jar (or eating them !). To do this completely non-destructively, you'd have to count the beans without taking them out of the jar....a difficult task indeed.

In the case where the object is temporarily destroyed during the computation and then restored at the end, this is considered to be ***Non-Destructive***. For instance, we may temporarily remove the jelly beans from the jar to count them and then put them back in when done. From the outsiders point of view, the jar arrangement has not been destroyed. We do realize however that the order of the jelly beans has been altered. If the order was important, then this process is considered destructive, as we are destroying the ordering.

Another approach that is common is to take the original object, **make a copy** and then write a destructive method on the copy. This is kind of like "cheating" but it does solve the problem, of course with the added running time overhead of copying the original object beforehand. This often leads to the need to write more than one method (i.e., use helper methods) to solve the problem. We will see later that this brings up the notion of ***indirect recursion***.

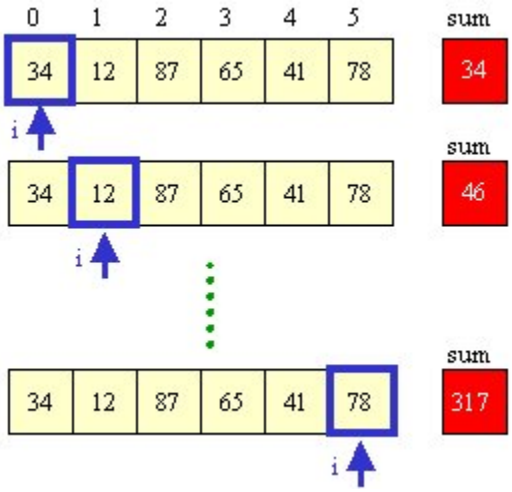
The notion of writing destructive or non-destructive methods is not something specific to recursion. In fact, you have already written some destructive AND non-destructive methods in COMP1405/1005.

Example (Summing Elements in an ArrayList)

Assume that we have an ArrayList of Integer objects. Consider writing a function that sums the integers in the ArrayList:

```
public static int sum(ArrayList<Integer>
anArrayList) {
    int sum=0;
    for(Integer x: anArrayList)
        sum += x;
    return sum;
}
```

As you can see, we simply go through the elements one by one and compute the total:



What about doing it recursively ? How do we define the problem recursively ?

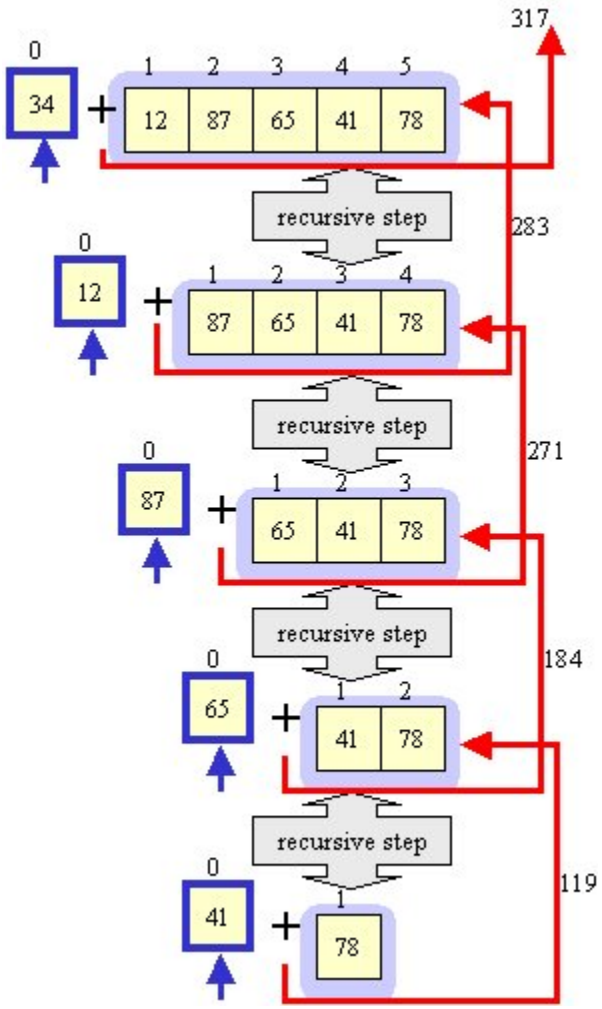
We can do this as follows:

- grab the first element and add it to the sum
- remove it from the ArrayList and recursively sum the remaining Integers.

Note this strategy is ***destructive*** in that it modifies the ArrayList by removing all of its elements. Nevertheless, let us examine how this is done.

First, we consider the recursion by finding the simple base cases:

- if the ArrayList is empty, then the sum is 0
- if the ArrayList has only one element, then the sum is that element
- otherwise we must do some recursion



Here is the code:

```
    public static int
sum(ArrayList<Integer> arrayList) {
        //BASE CASES
        if (arrayList.isEmpty())
           return 0;
        if (arrayList.size() == 1)
           return arrayList.get(0);

        //RECURSIVE STEP
        Integer element =
arrayList.get(0);
        arrayList.remove(element);
        return element.intValue() +
sum(arrayList);
    }
```

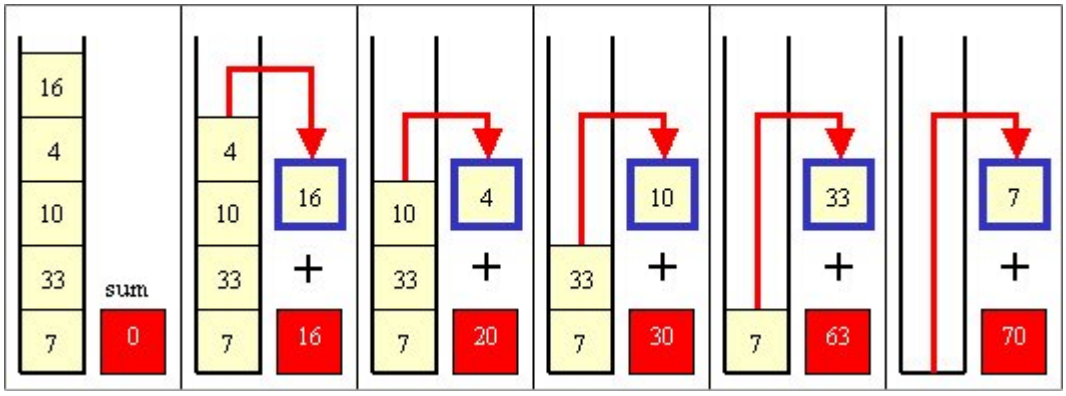
But wait. Think about what happens as we empty elements from the ArrayList and head towards the base case. Do we really need that second base case ? The answer is NO since one more step of the recursive case will bring us to the first base case. Hence, we can simplify the code by removing the second base case:

```
    public static int sum(ArrayList<Integer> arrayList) {
        if (arrayList.isEmpty()) return 0;

        Integer element = arrayList.get(0);
        arrayList.remove(element);
```

```
        return element.intValue() + sum(arrayList);
    }
```

This hardly seems more efficient than the non-recursive version!! As mentioned before, some problems are not meant to be done recursively. We are simply examining them recursively for practice in order to help us understand recursion.

In some cases, the recursive solution is simpler. Consider summing **Integer** objects that are in a **Stack**:

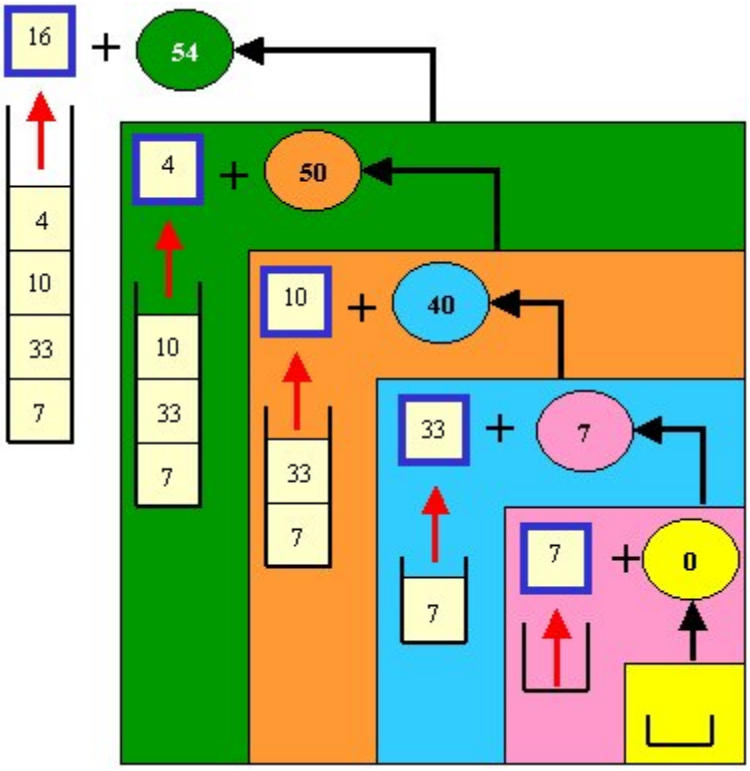



```
public static int sum(Stack<Integer> s) {
    int sum=0;
    while(!s.isEmpty())
        sum += s.pop();
    return sum;
}
```

Is this destructive ? Yes!

What does the recursive version look like ?

- The **sum**(s) = firstElementOf(s) + **sum**(s without the first element)



Here is the code:

```
public static int sum(Stack<Integer> s) {
    if (s.isEmpty()) return 0;
    return s.pop() + sum(s);
}
```

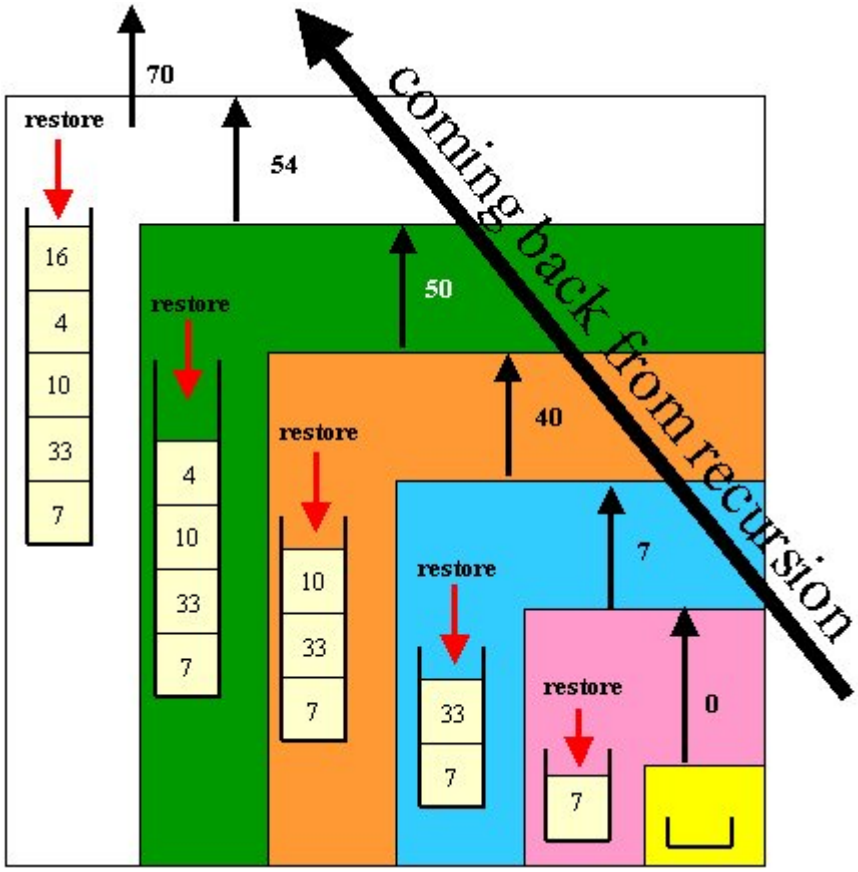
It is roughly the same amount of code. But now, what if we wanted to write these methods non-destructively ?
We need to put the items back onto the **Stack** so that the **Stack** is restored.
Here is the non-recursive, non-destructive version:

```
public static int sum(Stack<Integer> s) {
    int sum=0;
    Stack<Integer> tempStack = new Stack<Integer>();
    while(!s.isEmpty()) {
        Integer item = s.pop();
        sum += item;
        tempStack.push(item); // Keep backup of popped items
    }
    //Restore the original Stack
    while(!tempStack.isEmpty()) {
        s.push(tempStack.pop());
    }
    return sum;
}
```

Notice that we had to write code to keep track of the items that we removed so that we could put them back later. We used another **Stack** to keep this information, but we could have used any kind of collection.

Now what about the non-destructive recursive version ?

```
public static int sum(Stack<Integer> s) {
    if (s.isEmpty()) return 0;

    Integer element = s.pop();      // temporarily destroy stack
    int answer = element + sum(s);  // get recursive sum
    s.push(element);                // restore after the recursion
    return answer;
}
```



Hey! It is much simpler. We don't need extra variables. We simply restore the popped item on the way back from the recursion. This is an example of temporarily destroying the object in order to get an answer and then "undo"ing the destroying on the way back from the recursion. It is considered a non-destructive method. So we can see that the recursive solution is more elegant.

Example (Counting):

Consider counting the elements in some kind of collection. Perhaps we want to only count elements that satisfy some condition. In this example, we count the odd integers in an **ArrayList**. We will do it first destructively, and then alter our code to make it non-destructive.

Destructively, we can simply take off one element from the ArrayList each time through the recursion. We simply look at the number that we take out and add one to the total each time it is found to be odd.

Here is the destructive version inside a test class:

```
import java.io.*;
import java.util.*;

public class CountTest {

    public static int countOdd(ArrayList<Integer> nums) {
        //BASE CASE
        if (nums.isEmpty()) return 0;

        //RECURSIVE STEP
        Integer element = nums.get(0);
        nums.remove(element);

        if (element%2 == 0)
            return countOdd(nums);
        else
            return 1 + countOdd(nums);
    }

    public static void main(String[] args){
        int input = 0;
        ArrayList<Integer> nums = new ArrayList<Integer>();
        do {
            System.out.println("Enter integers one at a time
(0 to end):");
            if ((input = new Scanner(System.in).nextInt())
!= 0)
                nums.add(input);
        } while (input != 0);
        System.out.println("There were " + countOdd(nums) +
" odd integers entered.");
    }
}
```

How can we do it non-destructively ? We need to ensure that anything we remove is put back in

on the way back from the recursion. Since we begin by removing the first element from the ArrayList and then doing the recursion, we must add that removed element back to the front of the ArrayList AFTER the recursion is done. If we do this at every step, the ArrayList should be back to normal. Notice the changes that we need to make:

```
    public static int countOdd(ArrayList<Integer> nums) {
        //BASE CASE
        if (nums.isEmpty()) return 0;

        //RECURSIVE STEP
        Integer element = nums.get(0);
        nums.remove(element);

       int result = 0;
       if (element%2 == 0)
           result = countOdd(nums);
       else
           result = 1 + countOdd(nums);

       // Add the element back to the collection
    before returning
       nums.add(0, element);
       return result;
    }
```

In fact, we can simplify this code further, noticing that both conditions in the **IF** statement call recursively:

```
    public static int countOdd(ArrayList<Integer> nums) {
        //BASE CASE
        if (nums.isEmpty()) return 0;

        //RECURSIVE STEP
        Integer element = nums.get(0);
        nums.remove(element);

        int result = countOdd(nums);
        if (element%2 != 0) result++;

        // Add the element back to the collection before
returning
        nums.add(0, element);
        return result;
    }
```

5.5 Direct vs. Indirect Recursion

A method that calls a recursive method is considered ***recursive*** itself. If the method calls itself, it is considered to be ***directly recursive***. In general, it is possible for two or more methods to call one another repeatedly. This is commonly termed "***mutual***" recursion. For example, method **A** calls method **B**, which calls method **A** again etc...

We use the term "**indirect**" recursion to describe a method which itself is not recursive, but which calls a directly recursive method to compute its solution. The **isPalindrome()** method that we wrote is considered **directly** recursive as it calls itself recursively.

Example (Summing Integers in an Array)

Let us look again at the example of summing integers, but this time using an array of ints. We cannot remove from the array as we did with the **ArrayLists** and **Stacks** in the previous section of the notes. We must take a different, non-destructive, approach. How would we do this non-recursively ?

```
public static int sum(int[] theArray) {
    int result = 0;
    for (int i=0; i<theArray.length; i++)
        result += theArray[i];
    return result;
}
```

Now this is ***non-destructive*** because it merely accesses the array elements without removing or changing them.

How can we do this recursively ? At each step of the recursion, we will need to know which number we are adding, so we will also need an index. Where do we put this index ? It MUST be a parameter to the method so that this position needs to be carried through the recursive iterations:

```
private static int sum(int[] theArray, int n) {
    //BASE CASE: sum(theArray,0) = a[0]
    if (n == 0) return theArray[0];

    //RECURSIVE STEP: sum(theArray,n) = theArray[n] + sum(theArray,n-1)
    return (theArray[n] + sum(theArray,n-1));
}
```

Notice that this method adds the elements in reverse order. That is different from the non-recursive method. It is more natural this way since we continually reduce the value of **n** until we reach the base case of **n == 0**. Note that we did not do any error checking. What if n is greater than the array length ?

We do have one annoying problem now ... we have to supply an additional parameter.
To avoid this problem, we will make another method that will be called by the user in place of this one. This new method will provide the parameter for us:

```
    public static int sum(int[] theArray) {
        return (sum(theArray, theArray.length - 1);
    }
```

Can we have these two methods with the same name ? Yes ... since they have a different signature (i.e., parameter list). Recall that this is known as ***overloading***.

Now to test it we merely call this new method. In fact, we should make the original method **private**. This technique of making a kind of "wrapper" for the recursive method is known as ***indirect recursion***. The first method is ***directly recursive*** but this second one is called ***indirectly recursive*** since it does not call itself, but does call a recursive method. This second method is commonly called a ***helper*** method.

It should be a simple task for you to take the ArrayList method that we wrote earlier and make it non-destructive. Make sure you can do this. Here is the code altogether:

```
//Example of summing elements of Arrays and Arr using recursive and overloaded
sum methods.
import java.io.*;
import java.util.*;
public class SumTest {
    //This destructive method returns the sum of a vector of Integers
    public static int sum(ArrayList<Integer> arrayList) {
        //BASE CASE
        if (arrayList.isEmpty()) return 0;

        //RECURSIVE STEP
        Integer element = arrayList.get(0);
        arrayList.remove(element);
        return (element.intValue() + sum(arrayList));
    }

    //This is a helper method
    private static int sum(int[] theArray, int n) {
        if (n == 0) return theArray[0];
        return (theArray[n] + sum(theArray,n-1));
    }

    //This method returns the sum of an array of Integers.
    public static int sum(int[] anArray) {
        return (sum(anArray, anArray.length-1));
    }

    public static void main(String[] args) throws IOException {
        int MAX = 100, i=0, count=0, temp=0;
        int[] a = new int[MAX];
        ArrayList<Integer> arrayList = new ArrayList<Integer>();
```

```
        System.out.println("Enter the first integer (0 to end, Max=100):");

        //Add first number to array and vector
        temp = new Scanner(System.in).nextInt();
        a[i] = temp;
        arrayList.add(temp);

        //read numbers into array and vector
        while ((a[i] != 0) && (i < MAX)) {
            i++;
            count++;
            System.out.println("Enter another integer (0 to end):");
            temp = new Scanner(System.in).nextInt();
            a[i] = temp;
            arrayList.add(temp);
        }
        System.out.println("Here are the results:");
        System.out.println("Array Sum = " + sum(a));
        System.out.println("ArrayList Sum = " + sum(arrayList));
    }
}
```

Note that this method adds the final zero to the arrayList and the array. Can **you** fix this problem ? I hope so. Does it matter ?

Example (Reversing a StringBuffer):

Consider our earlier example in which we reversed the characters of a **String**. We could replace the **String** with a **StringBuffer** and then change our method so that the actual characters of the original **StringBuffer** would be reversed. This would be considered a destructive method since it modifies the original object parameters. Take notice of the method overloading and the use of a **private** helper method.

Our strategy will be to swap the first and last characters of the **StringBuffer** and then move inwards on the **StringBuffer** the same way we iterated through the **String** in our palindrome example. To do this, we will not shrink the **StringBuffer** each time. Instead, we will keep track of where we are by using indices as we traverse recursively. We will write the following method which will reverse the characters of the given **StringBuffer** within the indices specified by i and j:

```
private static void reverseBetween(StringBuffer s, int leftIndex, int
rightIndex) { ... }
```

We will call this method recursively, increasing **leftIndex** each time while decreasing **rightIndex**. Thus the base case will be the stopping condition when **leftIndex** is greater than or equal to the **rightIndex**. Here is the code:

```
    private static void reverseBetween(StringBuffer s, int leftIndex, int
rightIndex) {
```

```
        //BASE CASE
        if (leftIndex >= rightIndex )return;

        //RECURSIVE STEP
        char temp = s.charAt(leftIndex);
        s.setCharAt(leftIndex, s.charAt(rightIndex));
        s.setCharAt(rightIndex, temp);
        reverseBetween(s, leftIndex+1, rightIndex-1);
    }
```

Notice that the method does not return anything. It is simply a procedure that modifies the argument passed to it (i.e., modifies the **StringBuffer**).

Of course, this is not the method that we wanted. We want the following method signature:

```
     public static void reverse(StringBuffer s) { ... }
```

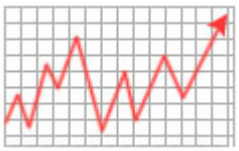
But we can make use of the **reverseBetween()** method, just ensuring that we pass in "good" **leftIndex** and **rightIndex** parameters:

```
    //Reverse the contents of StringBuffer s
    public static void reverse(StringBuffer s) {
        if (s.length() > 1)
           reverseBetween(s, 0, s.length()-1);
    }
```

Notice that this method is "indirectly" recursive since it calls a recursive method, but it does not call itself. Here is the test code:

```
public class DReverseTest {

    //Reverse the contents of StringBuffer s
    public static void reverse(StringBuffer s) { ... }

    private static void reverseBetween(StringBuffer s, int leftIndex, int
rightIndex) { ... }

    public static void main(String[] args) {
        //First check to see that there is at least one command line argument
        if (args.length == 0) {
            System.out.println("Usage: DReverseTest <aString>");
            System.exit(-1);
        }
        StringBuffer input = new StringBuffer(args[0]);
        System.out.println("string buffer's initial contents: " + input);

        reverse(input);  //Notice that we call the public method

        System.out.println("string buffer's final contents:   " + input);
    }
}
```

5.6 Some More Examples



Example (Averaging):

Consider finding the average of a set of Integers that are stored in an **ArrayList**. One approach is to use the **sum** method just mentioned and merely divide by the number of elements:

```
public static double avg(ArrayList<Integer> v) {
    return (sum(v) / (double)v.size());
}
```

This approach is fine and will work as desired. However, a more interesting challenge is to write a recursive averaging method, without using the **sum()** method. Let us try to see if we can express the averaging function recursively. We will say that avg(A, n) is the function for finding the average of **n** numbers stored in an ArrayList **A**. We will denote the elements of the ArrayList A to be A_1, A_2, A_3, ..., A_n.

What about the base cases ?

avg(A, 0) = 0
avg(A, 1) = A_1

Now the inductive case:

avg(A, n)
= (A_1 + A_2 + A_3 + ... + A_n) / n
= [(A_1 + A_2+ A_3 + ... + A_{n-1}) / n] + [A_n / n]
= [((n-1) / n) * (A_1 + A_2+ A_3 + ... + A_{n-1}) /(n-1)] + [A_n / n]
= [((n-1) / n) * avg(A, n-1)] + [A_n / n]
= [((n-1) * avg(A, n-1) + A_n] / n

So the recursive definition is:

avg(A, 0) = 0
avg(A, 1) = A_1
avg(A, n) = [((n-1) * avg(A, n-1) + A_n] / n

Now we can easily write the code:

```
import java.io.*;
import java.util.*;
public class AvgTest {
    //This method returns the average of n Integers in the given vector.
    private static double avg(ArrayList<Integer> a, int n) {
        //BASE CASE:
        if (n == 0) return 0;

        Integer last = a.get(a.size() - 1);
        if (n == 1) return last;

        //RECURSIVE STEP
        a.remove(last);
```

```
        return (((n-1) * avg(a, n-1) + last)/n);
    }

    //This method returns the average of an ArrayList of Integers
    public static double avg(ArrayList<Integer> a) {
        return (avg(a, a.size()));
    }

    public static void main(String[] args) {
         int input = 0;
         ArrayList<Integer> nums = new ArrayList<Integer>();
         do {
             System.out.println("Enter integers one at a time
(0 to end):");
             if ((input = new Scanner(System.in).nextInt())
!= 0)
                 nums.add(input);
         } while (input != 0);
         System.out.println("The average is " + avg(nums));
    }
}
```

Do we really need this extra index ? Can we re-write this method using direct recursion ? Well the index of **n** represents the size of the array, which changes as the recursion progresses. So we have to be careful:

```
 private static double avg(ArrayList<Integer> a) {
     if (a.isEmpty()) return 0;

     Integer last = a.get(a.size() - 1);
     if (a.size() == 1)
         return last;

     a.remove(last);
     int size = a.size();
     return (size * avg(a) + last)/(size+1);
 }
```

Notice the use of the **size** variable to that when the **ArrayList** size changes due to the recursion, this local variable stays constant. We can re-arrange the order of the calculations in the expression if we want to eliminate this extra **size** variable. We just have to make sure that the **size** is used BEFORE the recursive call:

```
private static double avg(ArrayList<Integer> a) {
    if (a.isEmpty()) return 0;

    Integer last = a.get(a.size() - 1);
    if (a.size() == 1)
        return last;
```

```
    a.remove(last);
    return (1/(a.size()+1.0))*(a.size() * avg(a) + last);
}
```

Of course, we can play games like this all day long :). We can actually eliminate the 2nd base case, since it is handled by the recursive call:

```
private static double avg(ArrayList<Integer> a) {
    if (a.isEmpty()) return 0;

    Integer last = a.get(a.size() - 1);
    a.remove(last);
    return (1/(a.size()+1.0))*(a.size() * avg(a) + last);
}
```

Also, we can get rid of the **last** variable by re-arranging the expression again:

```
private static double avg(ArrayList<Integer> a) {
    if (a.isEmpty()) return 0;
    return (1.0/(a.size()))*(a.remove(0) + a.size() * avg(a));
}
```

But WHY would you write such a complicated looking expression ??? It is better to leave the **last** variable as it was for ease of reading the code.

Example (Selecting):

Now, what if we wanted to not just count the odd integers in an **ArrayList**, but to gather and return them (i.e., select them) ? That is, make a new **ArrayList** that will contain all of the odd integers from the original ArrayList. Can we do this with direct recursion ? Perhaps we should make a helper method.

We can take the approach that is common to everyday life. Get a blank list ready, and write down all the odd integers in it. In terms of coding, we can prepare the blank "list" as an initially empty ArrayList passed in as a parameter. We can then add to this list as we go through the recursion. Hence we will need to pass this list along when we do the recursive calls.

```
import java.io.*;
import java.util.*;
public class SelectTest {

    //This destructive method returns the odd Integers in an ArrayList
    public static ArrayList<Integer> selectOdd(ArrayList<Integer> a) {
       return (selectOdd(a, new ArrayList<Integer>()));
    }

    //This is the helper method that does all of the work
    public static ArrayList<Integer> selectOdd(ArrayList<Integer> a, ArrayList<Integer>
result) {
```

```
        //BASE CASE:
        if (a.isEmpty()) return result;

        //RECURSIVE STEP
        Integer element = a.get(0);
        a.remove(element);

        if (element%2 != 0)
            result.add(element);

        return selectOdd(a, result);
    }

    public static void main(String[] args) {

         int input = 0;
         ArrayList<Integer> nums = new ArrayList<Integer>();
         do {
             System.out.println("Enter integers one at a time (0
to end):");
             if ((input = new Scanner(System.in).nextInt()) != 0)
                 nums.add(input);
         } while (input != 0);
         ArrayList<Integer> result = selectOdd(nums);
        System.out.println("Here are the odd integers:");
        for (int i=0; i<result.size(); i++) {
            System.out.println(result.get(i));
        }
    }
}
```

Can you write this method using direct recursion (i.e., no additional parameters) ?

```
    public static ArrayList<Integer> selectOdd(ArrayList<Integer> a) {
        if (a.isEmpty())
     return new ArrayList<Integer>();

        Integer element = a.get(0);
        a.remove(element);

        result = selectOdd(a);
        if (element%2 != 0)
            result.add(element);

        return result;
    }
```

Do the odd numbers come back in the same order as before ? Think about it.

What about making it non-destructive now:

```
    public static ArrayList<Integer> selectOdd(ArrayList<Integer> a) {
        if (a.isEmpty()) return new ArrayList<Integer>();

        Integer element = a.get(a.size()-1); //remove from end now

        a.remove(element);
        result = selectOdd(a);
        a.add(element); //restore after recursive call by adding to end

        if (element%2 != 0)
            result.add(element);

        return result;
    }
```

Example (Choosing k items from N):

Denote the number of ways of choosing **k** items out of **N** by **C(N,k)**. We can determine what this number is by doing an experiment and counting.
The approach we'll take is to look at any one particular item, say **X**, and decide if it is either going to be chosen in the answer or if it is not. All possibilities have to be considered.

- If **X** is chosen, there are **N-1** items left from which we must still choose **k-1** (one has already been chosen).
- If **X** is not chosen, there are **N-1** items left from which we still must choose **k** (since we haven't chosen any yet).

That is, **C(N,k) = C(N-1,k-1) + C(N-1,k)**

We can define the degenerate cases (the basis) as

1. **C(N,N) = 1**
2. **C(N,0) = 1**
3. **C(N,k) = 0** if **k > N** (We assume **k <= N** always)

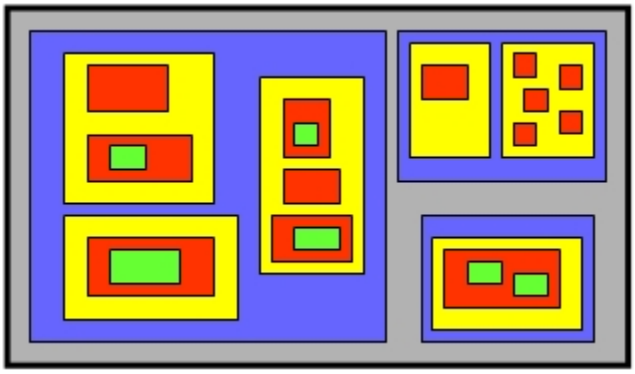
Here is the code:

```
//Computing C(n,k) example.  (use: java ChooseTest n k)
public class ChooseTest {
    //returns C(n,k)
    public static int choose(int n, int k) {
        //BASE CASES:
        if (n == k) return 1; //C(n,n) = 1
        if (k == 0) return 1; //C(n,0) = 1
        if (k > n) return 0;  //C(n,k) = 0
```

```
        //RECURSIVE CASES: C(N,k) = C(N-1,k-1) + C(N-1,k)
        return (choose(n-1,k-1) + choose(n-1,k));
    }

   public static void main(String[] args) {
        //First check to see that there is at least one command line argument
        if (args.length < 2) {
            System.out.println("Usage: ChooseTest <n> <k>");
            System.exit(-1);
        }
        int n = Integer.parseInt(args[0]);
        int k = Integer.parseInt(args[1]);

        System.out.println("C(" + n + "," + k + ")= " + choose(n,k));
    }
}
```

Example (Counting boxes within boxes):

Consider the following scenario. You wrap up your friends graduation gift in a box...but to be funny, you decide to wrap that box in a box and that one in yet another box. Also, to fool him/her you throw additional wrapped boxes inside the main box. The boxes-within-boxes scenario is recursive. Consider this problem now. We have boxes that are completely contained within other boxes and we would like to count how many boxes are completely contained within any given box. Here is an example where the outer box has 29 internal boxes:



Assume that there are some classes that implement the following **Box** interface:

```
public interface Box {
    public ArrayList<Box>   internalBoxes();      //Returns the boxes within
the receiver box
    public boolean          isEmpty();            //Return whether or not
there are any boxes within the receiver box
```

```
    public void             addBox(Box aBox);     //Add the given box to the
receiver
    public void             removeBox(Box aBox);  //Remove the given box from
the receiver
}
```

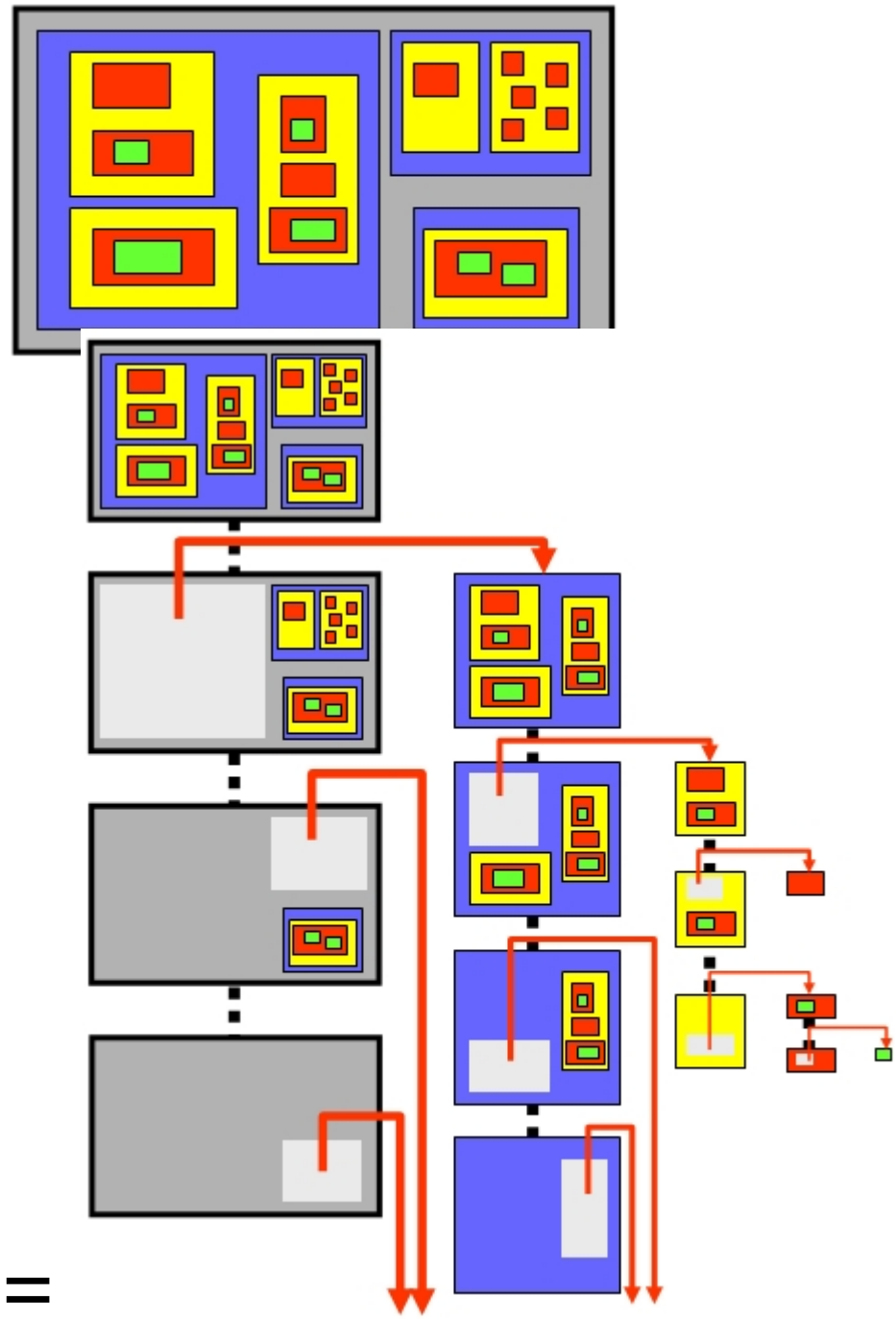
Now we have no idea how the boxes are stored or maintained. All we know is how to use the interface.
How can we write a recursive method to find the total number of boxes with a given box ?

- Base case: if the given box is empty, the answer is 0.
- Inductive case: the number of boxes directly within the receiver + the total of the number of internal boxes within each box that is within the receiver.

Here is the code:

```
public int countBoxes(Box aBox) {
    if (aBox.isEmpty())
        return 0;

    int count = 0;
    ArrayList<Box>   innerBoxes = aBox.internalBoxes();

    // Go through each internal box and get count
    // their internal boxes recursively
    for (Box b: innerBoxes) {
        count += countBoxes(b);
    }
    // Return the count of all internal boxes' boxes plus the number
    // of internal boxes at this level of the recursion
    return (count + innerBoxes.size());
}
```

Notice that we need the **for** loop here to go through all internal boxes. What if we were allowed to destroy the boxes ? Can we do this without a for loop ?



Here is the code:

```
public int countBoxes(Box aBox) {
    if (aBox.isEmpty())
        return 0;

    Box   anInnerBox = aBox.internalBoxes().get(0);
    aBox.removeBox(anInnerBox);
```

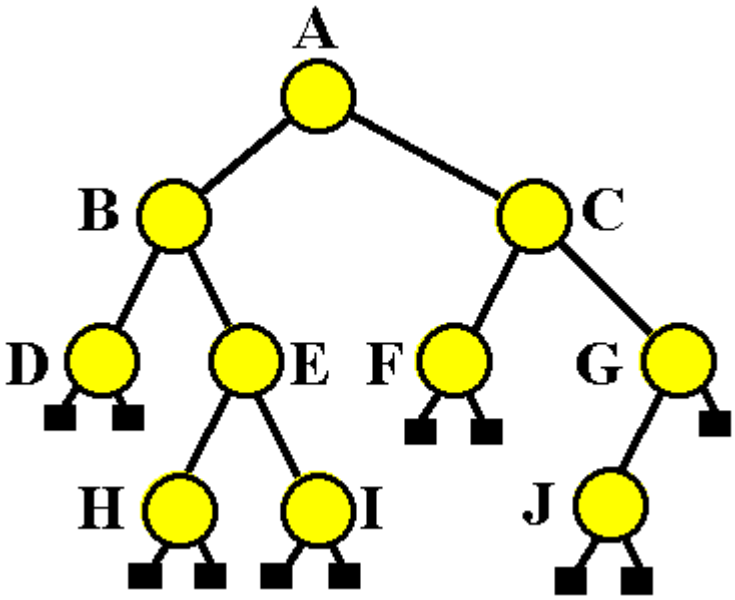
```
        return (countBoxes(aBox) + countBoxes(anInnerBox) + 1);
    }
```

Will this work ? Think about it. Are the recursive sub problems always smaller ? Which of the two pieces of code do you prefer ?
Try writing a class that implements the box interface and test this method out.

Example (Height of a Tree):

Consider a class called **BinaryTree** that keeps a collection of ***nodes***. The topmost node is the ***root*** of the tree. Each **node** stores an item of some kind and keeps pointers to a left and right ***child***. The children are themselves trees and are considered ***subtrees*** of the original tree. If a node has no left or right child, then **null** is stored there. A node with no children at all is considered a ***leaf***.

Here is an example of a binary tree:



A basic implementation of a binary tree may look like this:

```
public class TreeNode {
    private TreeNode  rightChild, leftChild;
    private Object    item;   // set to whatever the node represents

    public TreeNode(Object anObject) {
        this(anObject, null, null);
    }
```

```
    public TreeNode(Object anObject, TreeNode aLeftChild, TreeNode
aRightChild) {
        item = anObject;
        rightChild = aRightChild;
        leftChild = aLeftChild;
    }

    public final TreeNode rightChild() {
        return rightChild;
    }

    public final TreeNode leftChild() {
        return leftChild;
    }

    public final void setRightChild(TreeNode child) {
        rightChild = child;
    }

    public final void setLeftChild(TreeNode child) {
        leftChild = child;
    }
}
```

Notice that we called the class **TreeNode**. In fact, every node in the tree is a tree itself. It is actually a recursive data structure.

Here is an example of how to build the tree above:

```
public static void main(String args[]) {
    TreeNode root;
    root = new TreeNode("A",
                new TreeNode("B",
                            new TreeNode("D"),
                            new TreeNode("E",
                                        new TreeNode("H"),
                                        new TreeNode("I"))),
                new TreeNode("C",
                            new TreeNode("F"),
                            new TreeNode("G",
                                        new TreeNode("J"),
                                        null)));
}
```

The ***height*** of a tree is the number of nodes encountered along the longest path from (but not including) the root to a leaf of the tree.
The tree above has a height of 3. Can you write a recursive method that determines the height of a binary tree ?

```
      public int height() {
          if (leftChild == null)
              if (rightChild == null)
                 return 0;
```

```
            else
               return (1 + rightChild.height());
        else
            if (rightChild == null)
               return (1 + leftChild.height());
            else
               return (1 + Math.max(leftChild.height(),
    rightChild.height()));
    }
```

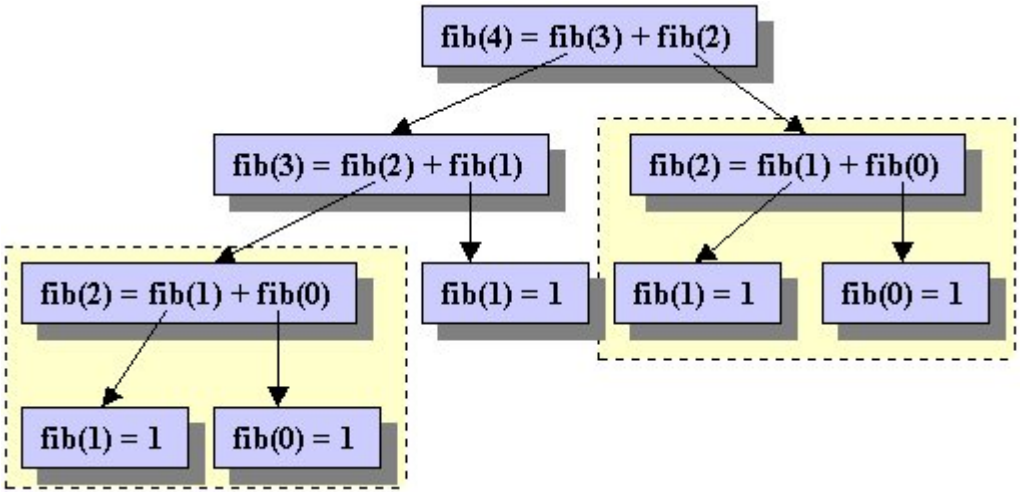
5.7 Efficiency with Recursion

Although recursion is a powerful problem solving tool, it has some drawbacks. A non-recursive (or iterative) method may be **more** efficient than a recursive one for two reasons:

- there is an overhead associated with large number of method invocations
- some algorithms are inherently inefficient.

Example: Computing the **n**th Fibonacci number can be written as:

1	if **n** = **0**
fib(n) = 1	if **n** = **1**
fib(n-1) + fib(n-2)	if **n** > **1**



Notice in the above computation some problems (e.g. **fibonacci(2)**) are being solved more than once, even though we presumably know the answer after doing it the first time. The following iterative solution avoids this. (It can also be avoided with a properly formulated recursive solution.)

```
public int finonacci(int n) {
    if (n < 0)
       return -1;

    int first = 1;
    int second = 1;
    int third = 1;
    for (i=2; i<=n; i++) {
        third = first + second; //compute new value
        first = second; second = third; //shift the others to the right
    }
    return third;
}
```

Try designing an efficient recursive Fibonacci method. You may use indirect recursion.

Here are some points to remember about efficiency:

- Anything that can be done with a loop using variables can be done recursively with the variables replaced by parameters.
- In some languages like **Lisp** and **Smalltalk**, recursion is normal. Consequently, recursive messages are implemented very efficiently.
- In Lisp, **for** loops and **while** loops are implemented recursively. Hence loops are no more (also no less) expensive than recursive invocations.

5.8 Practice Questions

Try writing recursive methods for the following problems:

Raising To A Power

Consider the evaluation of $\mathbf{x^n}$,where **n** is a non-negative integer.

- 1st Approach: (Could be done with linear recursion)

  $\mathbf{x^n = x * x * x*.....* x}$

- 2nd Approach (Recursive Formulation)

$x^{n/2} * x^{n/2}$ if **n** is even
$x^n = x * x^{n-1}$ if **n** is odd
$x^0 = 1$ if **n** is **0** (Basis Case)

Which approach would require fewer multiplications ?

Binary Search:

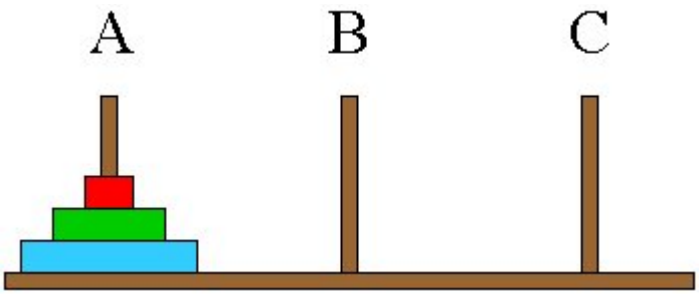
Write a recursive method which would locate an element in a **sorted** collection (i.e. Array or Vector) of **n** elements without having to check all the elements
(as in the worst case). That is, we'd like the maximum number of elements that need to be searched to be **log(n)**. The problem can be phrased recursively as follows.

- **Basis Case:** looking for an element in a collection of size 1 is easy - just check the single element.
- **Inductive Case:** Look at the middle element in the collection, if this bigger than the one you are looking for then search the portion of the collection before it. If it's smaller search the portion of the collection after it.

Notice how the problem of looking for an element in a collection of size **n**, is phrased in terms of looking for an element in a collection of size **n/2**. That's how we can be sure that only a logarithmic number of elements need to be examined.

A good exercise would be to build a **SortedCollection** class from scratch and use a binary search to implement its **contains(Object)** method (such as is implemented for class **Vector**).

The Tower of Hanoi Problem



The task is to move the disks from peg A to peg B using peg C as an intermediary (if necessary). The rules are as follows:

1. When a disk is moved, it must be placed on one of the three pegs.

2. Only one disk may be moved at a time, and it must be the top disk on one of the pegs.
3. A larger disk may never be placed on top of a smaller one.

A solution is best understood by considering the problem starting from simple cases to more complex cases.

Move (1 disk from A to B using C) => JUST DO IT

Move (2 disks from A to B using C) =>
    Move 1 A disk to C
    Move 1 A disk to B
    Move 1 C disk to B

Move (3 disks from A to B using C) =>
    Move 2 A disks to C using B (use previous step)
    Move 1 A disk to B
    Move 2 C disk to B using A

...

Move (n disks from A to B using C) =>
    Move n-1 A disks to C using B
    Move 1 A disk to B
    Move n-1 C disk to B using A

Implement a **TowersOfHanoi** application which will allow the user to specify some number of disks on one peg and then select which other peg they should be moved to. The application must then go and move the disks over to the new peg by following the rules above. This is a nice application to animate using a graphical user interface.
