Rotationally Monotone Polygons*

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Abstract

We introduce a generalization of monotonicity. An n-vertex polygon P is *rotationally monotone with respect to a point* r if there exists a partitioning of the boundary of P into exactly two polygonal chains, such that one chain can be rotated clockwise around r and the other chain can be rotated counterclockwise around r with neither chain intersecting the interior of the polygon. We present the following two results: (1) Given P and a center of rotation r in the plane, we determine in $O(n)$ time whether P is rotationally monotone with respect to r. (2) We can find all the points in the plane from which P is rotationally monotone in $O(n)$ time for convex polygons and in $O(n^2)$ time for simple polygons. We show that both algorithms are worst-case optimal by constructing a class of simple polygons with $\Omega(n^2)$ distinct valid centers of rotation.

A direct application of rotational monotonicity is the popular manufacturing technique of *clamshell casting*, where liquid is poured into a cast and the cast is removed by rotations once the liquid has hardened.

1 Introduction

Determining whether a polygon has certain properties, such as convexity, monotonicity, or star-shapedness, is a well-studied problem in computational geometry. This problem is not only important from a theoretical point of view, but also from a practical point of view. For surveys and application areas of classes of polygons, the reader is referred to the Handbook of Discrete and Computational Geometry [10, Chapter 23].

A polygon P is monotone in direction $\vec{d}$ if the intersection of P and any line in direction $\vec{d}$ is a convex set. Preparata and Supowit [15] determine in $O(n)$ time whether an n-vertex polygon is monotone. Rosenbloom and Rappaport [16] determine in $O(n)$ time whether a polygon P can be partitioned into exactly two monotone chains, where the two chains are monotone with different directions. Furthermore, they determine in $O(n \log n)$ time whether P can be decomposed into two monotone chains by cutting the boundary along a straight line. Dean et al. [7] introduce pseudo-star-shaped polygons. A polygon P is pseudo-star-shaped if there exists a point r, such that the interior of P is visible from r if one can see through single edges. ElGindy and Toussaint [8] consider radially monotone polygons. A polygon P is radially monotone if there exists a point r, such that every infinite half line emanating from r intersects P in a connected component. Note that the definitions of radially monotone and pseudo-star-shaped are equivalent.

*Research partially supported by NSERC. A preliminary version of this paper appeared at CCCG 2006 [4].

Further generalizations of polygons based on visibility were introduced. Avis and Toussaint [1] examine the visibility of a polygon P from one of the P's edges and present algorithms to determine visibility in linear time. The following three types of visibility are considered. First, the polygon P is completely visible from the edge e if every point on e is visible from every point in P. Second, the polygon P is strongly visible from the edge e if there exists a point on e that is visible from every point in P. Third, the polygon P is weakly visible from the edge e if every point in P is visible to any point on e. Bhattacharya et al. [2] consider weakly internally visible polygons. A polygon P is weakly internally visible from a line segment l completely contained in the interior of P if every point in P is visible to any point on l.

Toussaint [17] introduces a generalization of monotonicity in three dimensions. A polyhedron is weakly-monotonic if there exists a direction $\vec{d}$ such that the intersection of the polyhedron and any plane with normal $\vec{d}$ forms a simply-connected set. Bose and van Kreveld [5] give an algorithm to determine in $O(n \log n)$ time whether a simple n-vertex polyhedron is weakly-monotonic.

We introduce a new generalization of monotone polygons. A polygon P is *rotationally monotone* with respect to a point r in the plane if the boundary of P can be decomposed into exactly two polygonal chains, such that one chain can be rotated in clockwise orientation around r and the other chain can be rotated in counterclockwise orientation around r without either chain penetrating the interior of P. Two problems are addressed. First, given a center of rotation r in the plane, determine whether P is rotationally monotone with respect to r. We present a linear time algorithm to solve this problem. Second, an algorithm is presented to find all the points r in the plane, such that P is rotationally monotone with respect to r. The algorithm's running time for convex polygons is linear and for simple polygons is quadratic. We show that both algorithms are optimal in the worst case.

The notion of rotationally monotone polygons has a direct application to *clamshell casting*. Assume that we wish to manufacture an object modeled by a polyhedron P with combinatorial complexity n. Let the boundary of P be the *cast* of P. The polyhedron P is castable with respect to a line of rotation l if the cast of P can be partitioned into exactly two parts, such that one part can be rotated in clockwise orientation around l and the other part can be rotated in counterclockwise orientation around l without intersecting the interior of P or the cast of P. Bose et al. [3] use rotational monotonicity to solve the problem of clamshell casting in three dimensions.

This paper is organized as follows. Section 2 introduces the notation and preliminaries used throughout this paper. Section 3 discusses the problem of finding a partitioning of a given polygon based on a given point of rotation, and Section 4 discusses the problem of finding all of the points in the plane that allow a valid partitioning of the boundary of a polygon. Finally, Section 5 concludes and gives ideas for future work.

2 Preliminaries

Let P be a simple polygon in the plane with n vertices and let $int(P)$ and ∂P denote the interior and boundary of P, respectively, so that $P = int(P) \cup \partial P$. The edges of P are oriented counterclockwise so that $int(P)$ is located to their left. Parallel adjacent edges are not allowed, since this can be easily avoided by merging the two adjacent parallel edges. The aim is to determine whether the boundary of P can be partitioned into two pieces where each piece can be removed by a rotation. We specify below precisely what this means.

Definition 1. Let r and p be points in the plane. Denote the circular arc with center r and angle α starting at p winding in clockwise (cw) or counterclockwise (ccw) direction by $cwarc(r, p, \alpha)$ or

$ccwarc(r, p, \alpha)$ respectively. An edge e of P is *removable in cw orientation with respect to* r if

$$\exists\ \alpha > 0\ such\ that\ \forall\ p\ on\ e : cwarc(r, p, \alpha) \cap int(P) = \emptyset$$

and *removable in ccw orientation with respect to* r if

$$\exists\ \alpha > 0\ such\ that\ \forall\ p\ on\ e : ccwarc(r, p, \alpha) \cap int(P) = \emptyset.$$

Then, we call the cw or ccw orientation a *valid removal orientation for* e *with respect to* r respectively, and we call r a valid center of rotation for e. Figure 1 illustrates the definition of removability for edges.

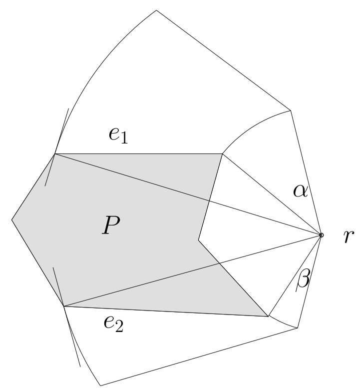


Figure 1: *The edges* e_1 *and* e_2 *are removable in cw orientation with angle* α *and ccw orientation with angle* β *with respect to* r *respectively.*

Definition 2. Let r be a point in the plane. A polygon P is *rotationally monotone with respect to* r, if ∂P can be partitioned into exactly two connected chains, such that all edges of one chain are removable in cw orientation with respect to r and all edges of the other chain are removable in ccw orientation with respect to r.

This implies that there exists an angle α, such that both chains can be rotated by angle α in cw or ccw orientation with respect to r, respectively, without colliding with each other. Note that the partitioning of the chain is not necessarily at vertices of P. We now outline a key property that characterizes all locations from which an edge is removable.

For an edge $e \in \partial P$ with incident vertices a and b, let $n_e(a)$ denote the line perpendicular to e passing through a. The line $n_e(a)$ divides the plane into two half planes and the open half plane containing b is denoted by $n_e^+(a)$ and the open half plane that does not contain b is denoted by $n_e^-(a)$. The supporting line $l(e)$ of e divides the plane into two half planes. Denote the open half plane located to the left of e when traversing P in ccw orientation by $l^+(e)$ and the open half plane located to the right of e when traversing P by $l^-(e)$, see Figure 2. The closure of an open set S is denoted by $cl(S)$.

Lemma 1. *Let* e *be an edge of* P *and denote the two vertices incident to* e *in ccw order by* a *and* b*. For the valid removal orientation of* e*, the following four cases are possible:*

1. *The edge* e *is removable using a cw rotation around* r*, if and only if* $r \in cl(n_e^-(a))$.

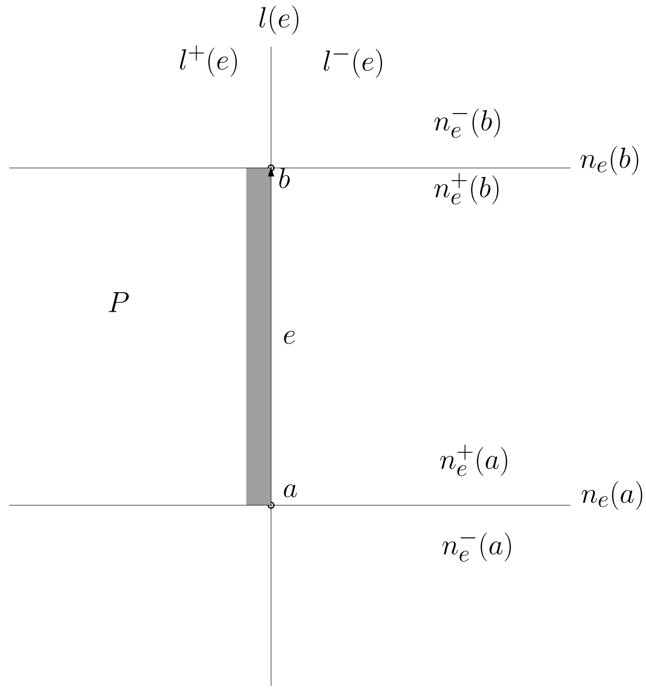


Figure 2: *The half planes associated with an edge e.*

2. *The edge e is removable using a ccw rotation around r, if and only if $r \in cl(n_e^-(b))$.*

3. *The edge e needs to be partitioned into two parts at the orthogonal projection of r on e in order to be removed, if and only if $r \in n_e^+(a) \cap n_e^+(b) \cap cl(l^-(e))$. One part of e is removable using a ccw rotation and the other one using a cw rotation around r. Let r^* be the orthogonal projection of r on e. Denote the edge with incident vertices a and r^* by e_1 and the edge with incident vertices r^* and b by e_2 respectively. The edge e_1 is removable using a ccw rotation around r and e_2 is removable using a cw rotation around r.*

4. *The edge e is not removable, if and only if $r \in n_e^+(a) \cap n_e^+(b) \cap l^+(e)$.*

Proof. Consider that every point p of e moves on $cwarc(r, p, \alpha)$ or $ccwarc(r, p, \alpha)$ when rotated by an angle α around r. Denote the vector from p to r by $\vec{pr}$ and the vector $\vec{pr}$ rotated in ccw orientation by 90^0 by $\vec{pr}^\perp$. The tangent of $cwarc(r, p, \alpha)$ or $ccwarc(r, p, \alpha)$ is $\vec{pr}^\perp$ or $-\vec{pr}^\perp$, respectively, for any point p. Denote by g the line passing through p in direction $\vec{pr}^\perp$ and denote by $g^+(p)$ the closed half plane bounded by g containing r. The two arcs $cwarc(r, p, \alpha)$ and $ccwarc(r, p, \alpha)$ are contained in $g^+(p)$.

Let p be an arbitrary point in the interior of e. There exists an open disk d with positive radius centered at p with the property that exactly half of d is contained in $int(P)$ and exactly half of d is contained in the exterior of P. Denote the ray starting at p propagating in direction $\vec{pr}^\perp$ by q^+ and denote the ray starting at p propagating in direction $-\vec{pr}^\perp$ by q^-.

Let $r \in cl(n_e^-(a))$ and let p be an arbitrary point in the interior of e. The intersection $d \cap q^+$ is located completely outside of $int(P)$. Hence, p can move by a small amount along $\vec{pr}^\perp$ without penetrating $int(P)$. Since $cwarc(r, p, \alpha) \subseteq g^+(p)$ and since $\vec{pr}^\perp$ is the tangent of $cwarc(r, p, \alpha)$

in p, small movements of p along $cwarc(r, p, \alpha)$ are possible without penetrating $int(P)$. Hence, $\exists\ \alpha > 0$ *such that* $\forall\ p$ *on* $e : cwarc(r, p, \alpha) \cap int(P) = \emptyset$. The intersection $d \cap q^-$ is completely contained in $int(P) \cup \{p\}$ and hence, p cannot move infinitesimally along $-\vec{pr}^{\perp}$ without penetrating $int(P)$. Since infinitesimal movements along $-\vec{pr}^{\perp}$ correspond to infinitesimal movements along $ccwarc(r, p, \alpha)$, there is no $\alpha > 0$ such that $\forall\ p\ on\ e : ccwarc(r, p, \alpha) \cap int(P) = \emptyset$. Hence, e is only removable using a cw rotation around r if $r \in cl(n_e^-(a))$.

Let $r \in cl(n_e^-(b))$ and let p be an arbitrary point in the interior of e. The intersection $d \cap q^-$ is located completely outside of $int(P)$. Hence, p can move by a small amount along $-\vec{pr}^{\perp}$ without penetrating $int(P)$. Since $ccwarc(r, p, \alpha) \subseteq g^+(p)$ and since $-\vec{pr}^{\perp}$ is the tangent of $ccwarc(r, p, \alpha)$ in p, small movements of p along $ccwarc(r, p, \alpha)$ are possible without penetrating $int(P)$. Hence, $\exists\ \alpha > 0$ *such that* $\forall\ p$ *on* $e : ccwarc(r, p, \alpha) \cap int(P) = \emptyset$. The intersection $d \cap q^+$ is completely contained in $int(P) \cup \{p\}$ and hence, p cannot move infinitesimally along $\vec{pr}^{\perp}$ without penetrating $int(P)$. Since infinitesimal movements along $\vec{pr}^{\perp}$ correspond to infinitesimal movements along $cwarc(r, p, \alpha)$, there is no $\alpha > 0$ such that $\forall\ p\ on\ e : cwarc(r, p, \alpha) \cap int(P) = \emptyset$. Hence, e is only removable using a ccw rotation around r if $r \in cl(n_e^-(b))$.

If $r \in n_e^+(a) \cap n_e^+(b) \cap cl(l^-(e))$, e is divided into two edges at the orthogonal projection r^* of r on e. Denote the edge with incident vertices a and r^* by e_1 and the edge with incident vertices r^* and b by e_2 respectively. As $r \in cl(n_{e_1}^-(r^*))$ and $r \in cl(n_{e_2}^-(r^*))$, e_1 is only removable using a ccw rotation around r and e_2 is only removable using a cw rotation around r.

If $r \in n_e^+(a) \cap n_e^+(b) \cap l^+(e)$, the orthogonal projection r^* of r on e cannot be rotationally removed. This means, there is no $\alpha > 0$ such that $cwarc(r, r^*, \alpha) \cap int(P) = \emptyset$ or $ccwarc(r, r^*, \alpha) \cap int(P) = \emptyset$ respectively. Therefore, e is not removable with respect to r.

This determines the removability of e depending on the location of r in the plane. Hence, the four statements of Lemma 1 follow directly. □

3 Decision Problem

In this section, we address the question of whether a polygon is rotationally monotone with respect to a given point of rotation and present an algorithm that solves the problem in linear time. The main idea is to examine the relationship between the removability of edges of P and the occurrence of local extrema of the distance between ∂P and r.

Assume that a polygon P and a center of rotation r are given. The aim is to determine whether P is rotationally monotone with respect to r. If P is rotationally monotone with respect to r, then the two points on ∂P, where the boundary of P is partitioned, need to be found.

Definition 3. A *near point c with respect to r* is defined as $c \in \partial P$ with the property that an arbitrarily small neighborhood of c on ∂P is completely outside of the open disk centered at r and passing through c. This means there exists a disk b centered at c with a positive radius, such that all points $q \in (\partial P \cap b) \setminus \{c\}$ are outside of the closed disk centered at r and passing through c.

Hence, if c is not a vertex, c is the orthogonal projection of r on an edge of P. Therefore, c locally minimizes the distance between the boundary of P and the center of rotation r.

Definition 4. A *far point f with respect to r* is defined as $f \in \partial P$ with the property that an arbitrarily small neighborhood of f on ∂P is completely contained in the closed disk centered at r and passing through f. This means there exists a disk b centered at f with a positive radius, such that all points $q \in \partial P \cap b$ are completely contained in the closed disk centered at r and passing through f.

A far point is always a vertex of P that locally maximizes the distance between the boundary of P and the center of rotation r.

Definition 5. Let $p \in \partial P$. If p is located in the interior of an edge e, split the edge into two edges at p. The valid removal orientation with respect to r is said to *change* at p if one of the edges incident to p is removable in cw orientation and the other edge incident to p is removable in ccw orientation with respect to r.

Lemma 2. *The valid removal orientation with respect to r changes at a point $p \in \partial P$ if and only if p is either a near point or a far point with respect to r.*

Proof. The proof consists of two parts. First, we show that the valid removal orientation with respect to r changes at $p \in \partial P$ if p is a near point or a far point with respect to r. At a far point f, an arbitrarily small neighborhood of f is completely contained in the closed disk induced by the circle b centered at r passing through f. Hence, there is a smaller circle concentric to b that passes through two points on ∂P located in an arbitrarily small neighborhood of f. As this circle intersects the polygon twice, one intersection point penetrates $int(P)$ when rotated infinitesimally in cw orientation with respect to r and the other intersection point penetrates $int(P)$ when rotated infinitesimally in ccw orientation with respect to r. Hence, it is not possible to remove the boundary of P in the same orientation. Hence, the valid removal orientation changes at f. The proof is similar for near points where b is enlarged by a small amount. Again, the two intersection points of the enlarged circle with the polygon can only be removed in different orientations with respect to r.

Second, the valid removal orientation with respect to r changes at no other point but a near point or a far point. Assume that the valid removal orientation with respect to r changes at $p \in \partial P$ with p neither a far point nor a near point. Hence, the circle b centered at r passing through p properly intersects P at p, since p neither locally maximizes nor locally minimizes the distance between ∂P and r. If p is not a vertex of P, but situated in the interior of an edge e of ∂P, e is split into two edges at p. Otherwise, p is a vertex of P and there exist exactly two edges adjacent to p. Therefore, the point p has two adjacent edges. As P is a simple polygon, locally it is located completely to the left of the boundary defined by the two edges adjacent to p. Hence, the valid removal orientation with respect to r does not change at p, which contradicts the initial assumption. Therefore, p must be either a near point or a far point for the valid removal orientation with respect to r to change. □

Theorem 1. *Given a center of rotation r, a polygon P is rotationally monotone with respect to r if and only if there exists exactly one near point c with respect to r and exactly one far point f with respect to r on ∂P.*

Proof. The proof consists of two parts. First, we show that P is rotationally monotone with respect to r if there exists exactly one near point c and exactly one far point f with respect to r. If there exists exactly one near point c and exactly one far point f with respect to r, the point c minimizes the distance between ∂P and r and f maximizes the distance between ∂P and r. Hence, P is completely contained in the closed annulus defined by the two concentric circles centered at r and passing through c and f respectively. The valid removal orientation with respect to r can only change at c and f (Lemma 2). Therefore, one part of the polygon can be removed using a cw rotation and the other part can be removed using a ccw rotation if P is cut at c and f.

Second, if P is rotationally monotone with respect to r then there exists exactly one far point and exactly one near point with respect to r. The boundary of a rotationally monotone polygon with respect to r is partitioned into two parts, i.e. there are exactly two points on ∂P where the valid removal orientation with respect to r changes. By Lemma 2, this implies that there are exactly two near or far points on ∂P. The extreme value theorem implies that there is always at least one local minimum and one local maximum with respect to the distance from r to ∂P [12, Chapter 3]. Therefore, there must exist at least one near point and one far point $\in \partial P$ with respect to r. Hence, there is exactly one near point and one far point with respect to r on a rotationally monotone polygon with respect to r. □

Theorem 1 allows us to determine whether a polygon is rotationally monotone given a center point r by testing how many points $p \in \partial P$ are local extrema with respect to the distance between p and r. The polygon is rotationally monotone if and only if there is exactly one maximum and one minimum. To do this test efficiently, we observe the following:

Observation 1. *For a simple polygon P and a point r in the plane, the number of points $c \in \partial P$ that locally minimize the distance between ∂P and r equals the number of points $f \in \partial P$ that locally maximize the distance between ∂P and r.*

Observation 1 holds because P is a simple closed polygon. Hence, the function describing the distance from r to ∂P is continuous and there is always a local minimum between two local maxima and vice versa for continuous functions [12, Chapter 3].

Hence, it is sufficient to consider local maxima to decide whether a polygon is rotationally monotone given a center point r. As each far point must be a vertex of P, one can test for multiple local maxima by traversing the polygon's vertices p and computing the distances between p and r. Furthermore, it is required to compute the minimal distance from r to each edge of P. The reason is that two consecutive vertices of P can both locally maximize the distance between r and ∂P. This takes linear time.

Theorem 2. *Given a polygon P with n vertices and a center of rotation r in the plane, we can test in $O(n)$ time whether P is rotationally monotone with respect to r.*

4 Determining all valid regions of rotational monotonicity

In this section, the aim is to find all points r in the plane, such that a given polygon is rotationally monotone with respect to r. We find those points by building an arrangement of lines in the plane and by identifying the regions of the arrangement containing points r with the property that P is rotationally monotone with respect to r.

Definition 6. The set of all points r in the plane with the property that P is rotationally monotone with respect to r is the *valid region of rotational monotonicity of P*. The complement of the valid region is the *invalid region for rotational monotonicity of P*.

The aim is to determine the valid region in the plane for a given polygon P by partitioning the plane into valid and invalid regions for rotational monotonicity. Once a query point r is given, it is possible to determine whether r is a valid center of rotation for P by determining whether r is contained in a valid or an invalid region of rotational monotonicity. We will see that convex polygons have a valid region that differs significantly from the valid region of non-convex simple polygons.

4.1 Rotational monotonicity of convex polygons

In this section, we consider convex polygons and show that it is possible to find the valid region of rotational monotonicity in linear time. The plane is partitioned into valid and invalid regions of rotational monotonicity by constructing the envelope of an arrangement of half lines.

Lemma 1 implies that every edge e with incident vertices a and b given in ccw order on ∂P splits the plane into regions of different valid removal orientations, see Figure 3.

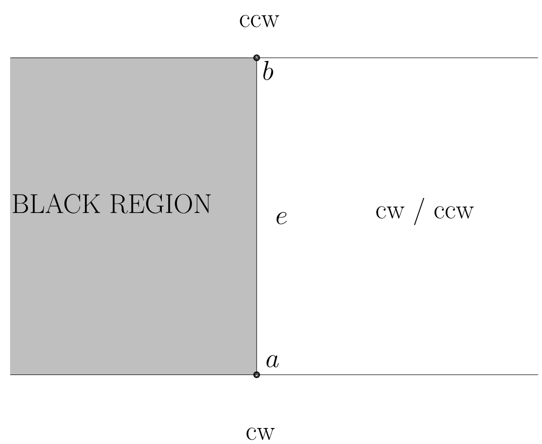


Figure 3: *An edge splits the plane into regions of different valid removal orientations*

Definition 7. Let e be an edge of P and denote the two vertices incident to e in ccw order by a and b. The open strip $n_e^+(a) \cap n_e^+(b) \cap l^+(e)$ is called the *black region of* e.

Note that the black region does not contain any valid centers of rotation r for which e is removable (see Lemma 1, case 4).

Lemma 3. *For a convex polygon P, $int(P)$ is contained in the union of the black regions of the edges of P.*

Proof. Every point $q \in int(P)$ has at least one near point $c \in \partial P$ with respect to q. As P is convex and as $q \in int(P)$, c is the orthogonal projection of q on an edge e and not a vertex of P. Hence, q is contained in the black region of e. □

Lemma 4. *A convex polygon P is rotationally monotone with respect to a center of rotation r if and only if r is not contained in the union of all black regions of edges of P.*

Proof. This proof consists of two parts. First, a convex polygon is not rotationally monotone with respect to r if r is contained in the union of all black regions of edges of P. If r is contained in the union of all black regions, it is contained in the black region of at least one edge e. The edge e is therefore not removable with respect to r by Lemma 1.

The second part is that P is rotationally monotone with respect to r if r is not contained in the union of all black regions of edges of P. Assume, r is outside of the union of the black regions, and P is not rotationally monotone. Theorem 1 and the Extreme Value Theorem [12, Chapter 3] imply that there are at least two far points with respect to r. Denote the two far points by f_1 and f_2. Two cases can occur: either $r \in int(P)$ or $r \notin int(P)$. If $r \in int(P)$, Lemma 3 ensures that P

is contained in the black region of at least one edge. Hence, $r \notin int(P)$ must hold. The following description is illustrated in Figure 4. Since $r \notin int(P)$, it is possible to compute two tangents from r to ∂P. Denote the two vertices where the tangents touch ∂P by t_1 and t_2, respectively. If a tangent touches ∂P in more than one vertex, choose the vertex closest to r as t_1 or t_2, respectively. The two tangents decompose ∂P into two chains, the lower chain contained in the triangle T with vertices t_1, t_2, and r and the upper chain not contained in T. Since P is convex, no far point of P with respect to r can be on the lower chain. Hence, both f_1 and f_2 are on the upper chain. There are two near points on ∂P with respect to r, one on each chain connecting f_1 and f_2. Since both f_1 and f_2 are on the upper chain, there must be a near point c_1 with respect to r on the upper chain between f_1 and f_2 (see Observation 1). Since P is convex and c_1 is on the upper chain, c_1 cannot be a vertex of P. Hence, c_1 is the orthogonal projection of r onto an edge e of P. Since e is on the upper chain and r projects orthogonally onto e, r is located to the left of e. Therefore $r \in n_e^+(a) \cap n_e^+(b) \cap l^+(e)$, where a and b denote the vertices incident to e. This means, r is contained in the black region of e. But this contradicts the initial assumption that r is not contained in the union of black regions of edges of P. Hence, P is only rotationally monotone with respect to r if r is outside of the union of black regions of edges of P. □

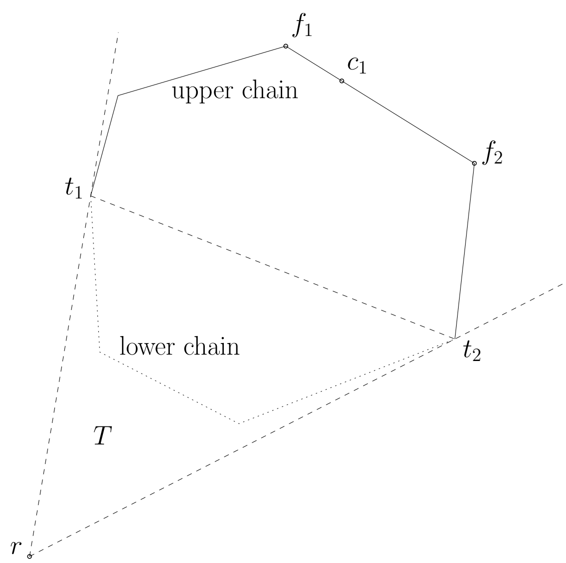


Figure 4: *The center r is located in the black region of the edge containing c_1.*

Lemma 5. *The valid region of rotational monotonicity of a convex polygon P consists only of unbounded regions in the plane.*

Proof. Note that Lemma 4 implies that the complement of the union of the black regions of edges of a convex polygon P is the valid region of rotational monotonicity of P. Assume there exists a point r in a bounded region such that P is rotationally monotone with respect to r. Then, r is contained in a region bounded by the black regions of at least two edges e_1 and e_2 of P and the convex polygonal chain h connecting e_1 and e_2 that has r to its left, see Figure 5. Let p_1 and p_2 be the vertices $e_1 \cap h$ and $e_2 \cap h$. The vertices p_1 and p_2 minimize the distance from r to e_1 and e_2 respectively. As the function describing the distance from r to ∂P is continuous and as P is simply connected, there exists at least one near point c with respect to r on h. As r is located to the left of h and as h is convex, c is located in the interior of an edge e with incident vertices a and b. Hence, $r \in n_e^+(a) \cap n_e^+(b) \cap l^+(e)$, i.e. r is contained in the black region of e. This contradicts the initial assumption and proves that the valid region of rotational monotonicity of P consists only of unbounded regions in the plane. $\square$

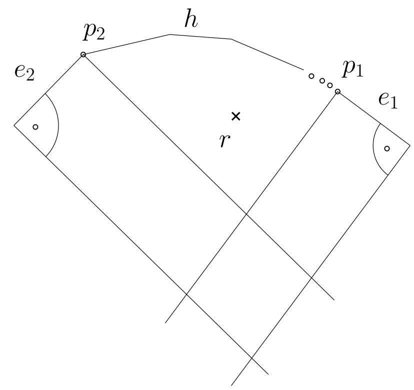


Figure 5: *Location of a point r in a bounded region.*

Based on Lemma 4 and Lemma 5, we compute the boundary of the union of all black regions of edges of P. For this, the notion of an envelope of n lines is defined.

Definition 8. A set of n lines in the plane induces a subdivision S of the plane. The *envelope of the n lines* is the polygon formed by the bounded edges of all the unbounded regions of S [11].

Similarly, a convex polygon P and the half lines bounding the black regions of its edges induce a subdivision S of the plane. Parallel half lines with the same orientation intersect at infinity and are therefore considered to be bounded edges. The polygon formed by the bounded edges of all the unbounded regions of S is called the *envelope of the arrangement induced by P*.

Lemma 5 implies that all valid regions of rotational monotonicity of P are contained in the complement of the envelope of the arrangement induced by P. This can be computed in linear time by modifying the algorithm by Keil [11] for computing envelopes of arrangements of lines as outlined below.

Theorem 3. *Given a convex polygon P with n vertices, a description of the valid region of rotational monotonicity of P has $O(n)$ size and can be computed in $O(n)$ time.*

Proof. Using the algorithm of Keil [11], it is possible to compute the envelope of an arrangement of n lines in $O(n)$ time given that the lines are sorted according to their slope. Keil's algorithm processes the set of lines in order of slope and uses a stack to maintain intermediate results for the envelope. This algorithm can be modified to find the union of all black regions of edges of P by defining an arrangement consisting of the half lines that bound black regions of edges. In this arrangement, the left and the right envelopes are computed, and their union corresponds to the union of all black regions of P. The modified algorithm first splits the polygon at the two points with minimum and maximum y-coordinate. The right envelope is computed by starting at the lowest point of the polygon and traversing it in clockwise order up to the highest point. For each edge e we traverse, denote the half line in the direction of the inner normal of e passing through the first vertex of e encountered during the traversal by l_i and the half line in the direction of the inner normal of e passing through the second vertex of e encountered during the traversal by l_i^*, $1 \leq i \leq s, s < n$. See Figure 6. Denote by B_i the convex polygonal chain bounding the region below the half lines l_1 to l_i, $1 \leq i \leq s$, and by A_i the convex polygonal chain bounding the region above the lines l_{i+1}^* to l_s^*, $0 \leq i \leq s-1$. Concatenate A_0, for $1 \leq i \leq s-1$ the boundary of $A_i \cap B_i$, B_s, and in case that A_0 and B_s are disjoint the part of P used to compute the right envelope. For a visualization of the result of this right envelope, refer to Figure 6.

To compute the left envelope, traverse the polygon in ccw direction starting at the lowest point and ending at the highest point. Define l_i and l_i^*, $1 \leq i \leq s, s < n$ as above for every edge of P. Computing A_i and B_i in the same way as before and concatenating A_0, for $1 \leq i \leq s-1$ the boundary of $A_i \cap B_i$, B_s, and in case that A_0 and B_s are disjoint the part of P used to compute the left envelope yields the left envelope. Note that the only difference between this algorithm and Keil's algorithm is the use of two different sets of lines l_i and l_i^* to compute B_i and A_i, respectively. Hence, only minor changes in Keil's algorithm are required to perform these computations. As there are $2n$ half lines already sorted by slope, this algorithm takes $O(n)$ time.

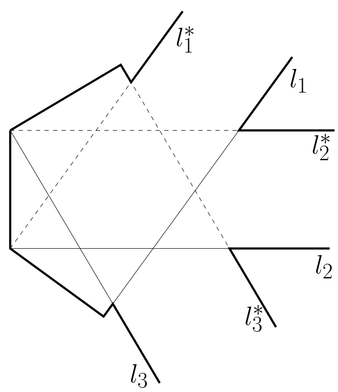


Figure 6: *Result of the right envelope algorithm (shown in bold)*

Two planar regions are created, and if we imagine that parallel lines intersect at infinity, the two regions are simply connected planar polygons. The algorithm by Finke and Hinrichs [9], that computes the overlay of simply connected planar subdivisions in time linear in the size of the output, is used to compute the union of those two regions. The algorithm assumes that the two subdivisions are given in quad view data structure and changes that structure in a way that the result represents the overlay of the two regions.

The size of the two envelopes E_1 and E_2 is linear in the number n of vertices of the polygon P, because it can be computed using Keil's algorithm in $O(n)$ time. As both envelopes ordered in clockwise order are given, one can construct a quad view data structure in linear time. The time required for Finke and Hinrichs's algorithm is $O(n+k)$, where n is the combined size of the two polygons to be overlayed and k is the number of intersection points of E_1 and E_2. Lemma 5 guarantees that there are no unbounded valid regions in the overlay of E_1 and E_2. Hence, when an edge of E_1 intersects an edge of E_2, only one of the edges can have further intersection points with E_1 or E_2 respectively. Therefore, the number of intersection points of E_1 and E_2 is $O(n)$ resulting in an $O(n)$ time algorithm. In the resulting subdivision, any region labeled as unbounded is a valid region of rotational monotonicity of P.

The combination of the two algorithms allows to find the valid region of rotational monotonicity of P in $O(n)$ time where n is the number of vertices of P. □

Corollary 1. *A convex polygon P with n vertices can be preprocessed in $O(n)$ time, such that for any given point r, we can decide in $O(\log n)$ time if P is rotationally monotone with respect to r.*

Proof. Theorem 3 allows to find the valid region of rotational monotonicity of P in $O(n)$ time. Hence, in $O(n)$ time, the plane is preprocessed, such that every face of the planar subdivision induced by black regions of P is labeled as a valid or invalid region.

For any query point r, after $O(n)$ preprocessing time, it is possible to determine the face of the arrangement containing r in time $O(\log n)$ [13]. Once the face is known, we can determine in constant time whether that face is contained in the union of the black regions of P, i.e. whether r is a valid center of rotation. □

4.2 Rotational monotonicity of simple polygons

In this section, we consider simple (not necessarily convex) polygons with n vertices and show that it is possible to find the valid region of rotational monotonicity of P in $O(n^2)$ time. If the aim is to report the valid region, this time bound is shown to be worst case optimal.

Let r be a point in the plane. If the valid removal orientation of a simple polygon P changes with respect to r at a reflex vertex $v \in \partial P$, v penetrates $int(P)$ when rotated infinitesimally around r with arbitrary orientation. This yields the following observation:

Observation 2. *A rotationally monotone polygon P with respect to r cannot be divided at one of its reflex vertices v unless the center of rotation r is v. Hence, v cannot be a far point with respect to r and v can only be a near point with respect to r if $r = v$.*

Definition 9. Let v be a vertex of P and denote the two edges adjacent to v by e_1 and e_2. The *near cone of* v is defined as $cl(n^-_{e_1}(v) \cap n^-_{e_2}(v))$ and denoted by $NC(v)$.

The near cone of v is the set of all points $X \in \mathbb{R}^2$ with the property that v is a near point with respect to X, see Figure 7.

Definition 10. Let v be a vertex of P and denote the two edges adjacent to v by e_1 and e_2. The *far cone of* v is defined as $n^+_{e_1}(v) \cap n^+_{e_2}(v)$ and denoted by $FC(v)$.

The far cone of v is the set of all points $X \in \mathbb{R}^2$ with the property that v is a far point with respect to X, see Figure 7.

Definition 11. The *black region of a reflex vertex* v is $(NC(v) \cup FC(v)) \setminus \{v\}$.

Note that Observation 2 ensures that the black region of v does not contain any valid centers of rotation r that allow v to be removed from ∂P.

Lemma 6. *For a simple polygon P, $int(P)$ is contained in the union of the black regions of the edges and the reflex vertices of P.*

Proof. Every point $p \in int(P)$ has at least one near point $c \in \partial P$ with respect to p. If c is the orthogonal projection of p on the interior of an edge e, p is contained in the black region of e. Otherwise, c is a reflex vertex and p is contained in the black region of c. □

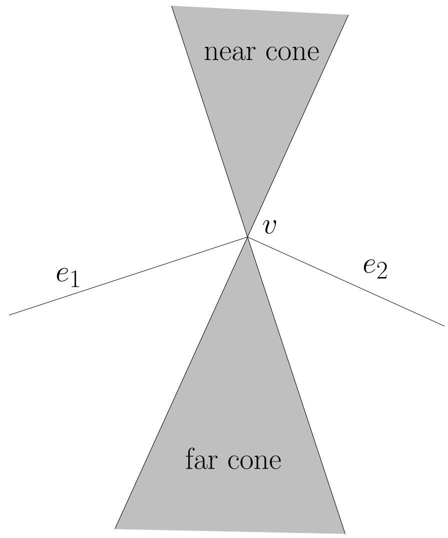


Figure 7: *The near cone and the far cone of v.*

Lemma 7. *A simple polygon P is rotationally monotone with respect to a center of rotation r if and only if r is not contained in the union of all black regions of edges and reflex vertices of P.*

Proof. This proof consists of two parts. First, a simple polygon is not rotationally monotone with respect to r if r is contained in the union of all black regions of edges and reflex vertices of P. If r is contained in the union of all black regions, it is either contained in the black region of at least one edge e or in the black region of at least one reflex vertex v. Hence, either e or v cannot be removed from ∂P.

Second, a simple polygon is always rotationally monotone if r is not contained in the union of the black regions of its edges and reflex vertices. Assume that P is not rotationally monotone with respect to r and that r is not contained in the union of black regions of edges and reflex vertices of P. Hence, there are at least two far points f_1 and f_2 on ∂P with respect to r, see Theorem 1 and the Extreme Value Theorem [12, Chapter 3]. Note that neither f_1 nor f_2 can be a reflex vertex as r is not contained in the black region of any reflex vertex. Two situations are possible: either $r \in int(P)$ or $r \notin int(P)$. Lemma 6 ensures that $r \notin int(P)$ as any point $q \in int(P)$ is contained in the union of the black regions of the edges and reflex vertices of P. The following description is illustrated in Figure 8. Denote the far point with smallest distance to r by f_1. If this far point is not unique, choose an arbitrary far point with smallest distance to r. Denote the circle centered at r passing through f_1 by c. In a local neighborhood of f_1, ∂P is contained in the interior of c. However, since f_2 is a far point on ∂P with respect to r with greater or equal distance from r than f_1, ∂P intersects c in at least one point not equal to f_1. Find the first point q_1 of ∂P that intersects c when starting at f_1 and walking along ∂P in ccw orientation. The polygonal chain starting at f_1 and ending at q_1 splits c into two regions. If r is contained in the region of c located to the left of the polygonal chain starting at f_1 and ending at q_1, we call the polygonal chain an *upper chain*. Otherwise, find the first point q_2 of ∂P that intersects c when starting at f_1 and walking along ∂P in cw orientation. By the Jordan Curve Theorem [14, Chapter 1], the polygonal chain starting at q_2 and ending at f_1 must be completely contained in the region of c located to the left of the polygonal chain starting at f_1 and ending at q_1. Furthermore, $int(P)$ is contained in the region bounded by the polygonal chain starting at q_2 and ending at f_1 and by the polygonal chain starting at f_1 and ending at q_1. Hence, r is contained in the region of c located to the left of the polygonal chain starting at q_2 and ending at f_1. Denote the polygonal chain starting at q_2 and ending at f_1 by *upper chain*. The points f_1, q_1, and q_2 are points that maximize the distance from the two polygonal chains considered above to r. Hence, by Observation 1 there exists a near point c_1 on the upper chain. Since r is located in the region of c located to the left of the upper chain, c_1 cannot be a convex vertex. Hence, c_1 is either located on an edge e of P or c_1 is a reflex vertex of P. If c_1 is located on an edge e, c_1 is the orthogonal projection of r on e and therefore, r is contained in the black region of e. Otherwise, c_1 is a reflex vertex that is a near point and therefore, r is contained in the black region of c_1. Hence, r is either contained in the black region of the reflex vertex c_1 or in the black region of the edge e. But this contradicts the initial assumption that r is not contained in the union of all black regions of edges and reflex vertices of P. Hence, P is only rotationally monotone with respect to r if r is outside of the union of the black regions of the edges and reflex vertices of P. □

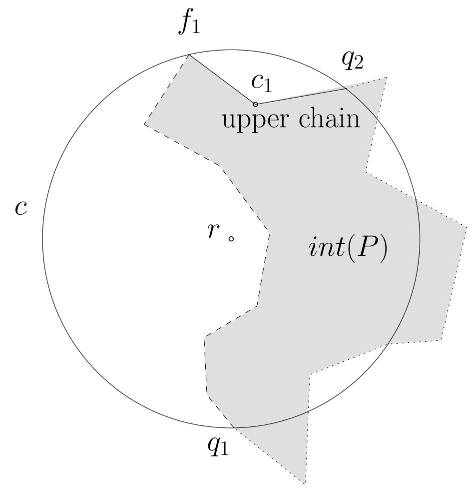


Figure 8: *The center r is located in the black region of the reflex vertex c_1.*

Theorem 4. *Given a simple polygon P with n vertices, a description of the valid region of rotational monotonicity of P has $O(n^2)$ size and can be computed in $O(n^2)$ time.*

Proof. We preprocess the plane by constructing the full arrangement A of the (full) lines bounding the black regions of edges and reflex vertices. A doubly-connected edge list of the arrangement of n lines has complexity $O(n^2)$ and can be constructed in $O(n^2)$ time. See [6, Chapter 8]. Once A is constructed, each face needs to be labeled as valid or invalid region of rotational monotonicity. For this purpose, a boolean value is associated with every edge e and reflex vertex v of P that indicates whether the current location is contained in the black region of e or v respectively. We start at an arbitrary face f of A and test for each edge and reflex vertex of P whether it causes f to be invalid. After testing, we set the boolean value of each edge and reflex vertex appropriately and compute the number b of edges and reflex vertices that cause f to be invalid. Clearly, f is valid if and only if $b = 0$. This computation takes $O(n)$ time as every edge and reflex vertex of P needs to be considered. Next, A is traversed in depth-first order on the graph induced by the vertices and the edges of A. Each time, an edge e_A of A is crossed, we update both the boolean value of the edge or reflex vertex of P that induces e_A and the counter b. This way, every face of A is labeled in constant time per face. The edge e_A and its incident vertices are valid regions of rotational monotonicity if and only if one or more of e_A's adjacent faces is a valid region of rotational monotonicity. Hence, A can be labeled in $O(n^2)$ time. □

Corollary 2. *A simple polygon P with n vertices can be preprocessed in $O(n^2)$ time, such that for any query point r, we can decide in $O(\log n)$ time if P is rotationally monotone with respect to r.*

Proof. Theorem 4 allows to find the valid region of rotational monotonicity of P in $O(n^2)$ time. Hence, the plane is preprocessed, such that every face of the planar subdivision induced by black regions of P is labeled as valid or invalid region in time $O(n^2)$.

For any query point r, after $O(n^2)$ preprocessing time, it is possible to determine the face of the arrangement containing r in time $O(\log n)$ [13]. Once the face is known, the label of the face can be retrieved in constant time. Hence, determining whether r is a valid center of rotation for P takes $O(\log n)$ time. □

We now examine the worst case complexity of the valid region of rotational monotonicity of P. In the best case, i.e. in the case of a convex polygon, the number of valid regions of rotational monotonicity is $O(n)$. The number of valid regions cannot be $\omega(n^2)$ as the complexity of an arrangement induced by $O(n)$ lines is $O(n^2)$. Next we show that there exists a class of simple polygons where the number of valid regions is $\Omega(n^2)$. This implies that the $O(n^2)$ time bound is worst case optimal if the aim is to report all valid regions of rotational monotonicity for a simple polygon. We now outline the construction of the lower bound.

We construct a simple polygon P consisting of $n = 3s - 1$ vertices located on two different polygonal chains. Let s vertices of P be evenly distributed on the upper half of the unit circle. The coordinates of those vertices are

$$(\cos((i-1)\phi_1), \sin((i-1)\phi_1)), \quad i = 1, \dots, s,$$

where $\phi_1 = \frac{\pi}{s-1}$. Hence, the vertices form a convex polygonal chain c_1. All valid regions induced by c_1 are cones whose apexes a are on the unit circle and opening angle $\frac{\phi_1}{2}$, see Figure 9.

The second polygonal chain c_2 consists of $2s-1$ vertices. Let s vertices of c_2 be evenly distributed on the arc of the circle with center $\left(-\frac{1}{2}, 0\right)$ and radius 1 starting at $\frac{3\pi}{2}$ and ending at $\frac{25\pi}{16}$. The coordinates of those vertices are

$$\left(-\frac{1}{2} + \cos\left(\frac{3\pi}{2} + (i-1)\phi_2\right), \sin\left(\frac{3\pi}{2} + (i-1)\phi_2\right)\right), \quad i = 1, \dots, s,$$

where $\phi_2 = \frac{\pi}{16(s-1)}$. Denote the vertices by $v_1, \dots, v_s$ and note that v_i is not located in the interior of the unit disk for $i = 1, \dots, s$. Define the vertices v_0, v_{s+1} as

$$\left(-\frac{1}{2} + \cos\left(\frac{3\pi}{2} - \phi_2\right), \sin\left(\frac{3\pi}{2} - \phi_2\right)\right), \left(-\frac{1}{2} + \cos\left(\frac{3\pi}{2} + s\phi_2\right), \sin\left(\frac{3\pi}{2} + s\phi_2\right)\right),$$

respectively. Let $s - 1$ vertices of c_2 be defined as the intersections of the line passing through v_{i-1} and v_i with the line passing through v_{i+1} and v_{i+2}, where $i = 1, \dots, s-1$. These vertices are located on a circle. The polygonal chain c_2 consists of $s - 2$ reflex, $s - 1$ convex, and 2 boundary vertices. Note that c_2 consists of sides of isosceles triangles, i.e. all the edges have the same length, see Figure 10. Valid regions bounded by part of c_2 and two parallel half lines occur.

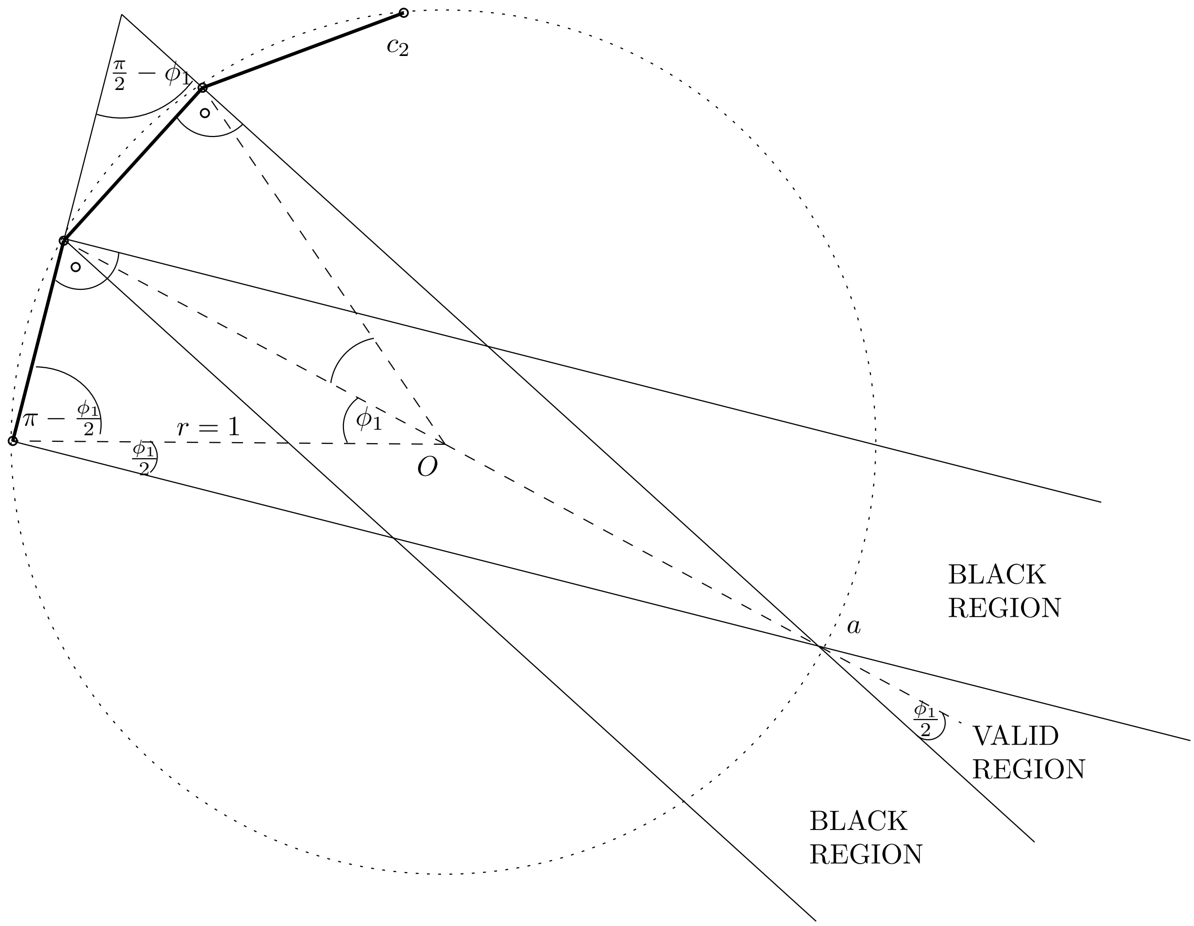


Figure 9: *Approximation of half circle*

The two polygonal chains c_1 and c_2 can now be connected by two edges. This does not introduce further reflex vertices to P, but only two black regions of the new edges. Those black regions have no influence on further considerations. Each of the black regions induced by reflex vertices on c_2 induces a bounded valid region when intersecting the valid region induced by vertices located on the arc of c_1 starting at $\frac{7\pi}{16}$ and ending at $\frac{\pi}{2}$. Hence, there are at least $(s-2)\lfloor\frac{s}{8}\rfloor$ bounded valid regions. As $n = 3s - 1$, there are $\frac{n-5}{3}\lfloor\frac{n+1}{24}\rfloor = \Omega(n^2)$ bounded valid regions. Hence, the number of valid regions of rotational monotonicity of a simple polygon is $\Omega(n^2)$. An example with $s = 10$ is shown in Figure 11.

5 Conclusion and Future Work

We have introduced the notion of rotationally monotone polygons and we have studied the problem of rotational monotonicity in two dimensions. An algorithm was developed to solve the problem of determining whether a polygon with n vertices is rotationally monotone with respect to a given point in the plane with running time $O(n)$. Furthermore, two algorithms were developed to report all the valid centers of rotation for a given polygon in the plane. The running times of the algorithms

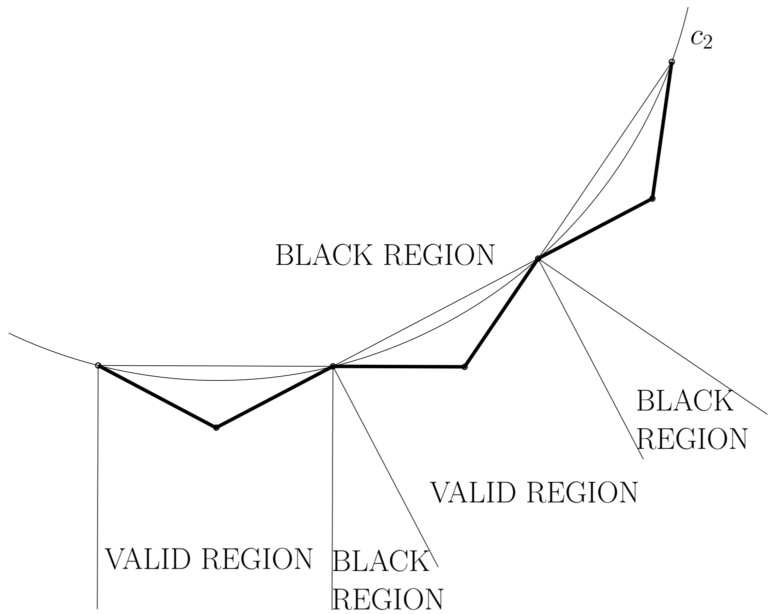


Figure 10: *Polygonal chain*

are $O(n)$ for convex polygons and $O(n^2)$ for simple polygons in general and shown to be worst-case optimal.

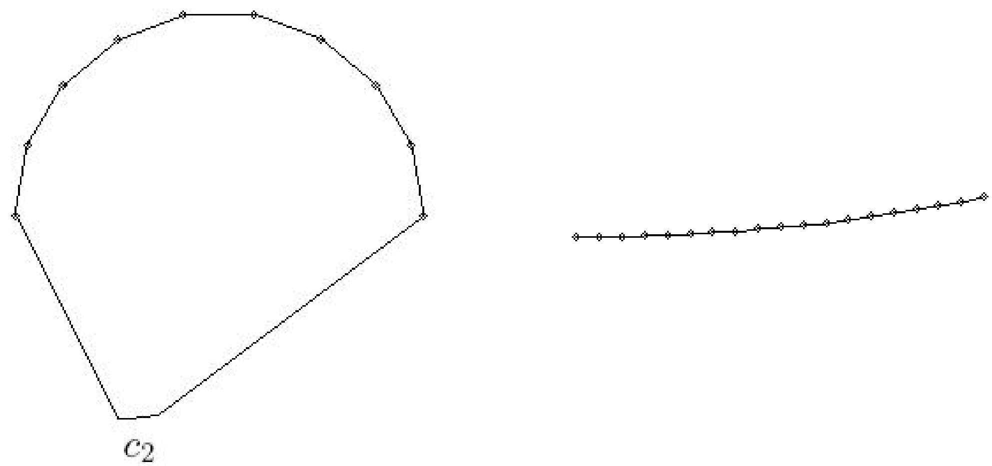
The new results on rotationally monotone polygons have direct applications to clamshell casting in three dimensions [3].

The following interesting related problems require further research. The definition of rotational monotonicity with respect to a point r only tests whether the boundary of the polygon can be decomposed into two chains, such that both chains can be rotated around r by an arbitrarily small angle without colliding with the interior of the polygon. An interesting extension is to determine whether the two chains can be rotated by a given angle α without colliding with the interior of the polygon. Another related problem is to find the maximal angle α the two chains can be rotated by without colliding with the interior of the polygon for a rotationally monotone polygon with respect to r.

For simple polygons, we showed the running time $O(n^2)$ to be worst-case optimal if the aim is to report all the valid centers of rotation. It remains an open problem whether it can be determined faster whether there exists a point r in the plane, such that a simple polygon is rotationally monotone with respect to r.

6 Acknowledgments

S. Wuhrer would like to thank the members of the Computational Geometry group at Carleton University for their generous financial support.



(a) (b)

Figure 11: *Example with $s = 10$. (a) shows the polygon, (b) shows an enlargement of the polygonal chain c_2.*

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