Strong Matching of Points with Geometric Shapes*

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Abstract

Let P be a set of n points in general position in the plane. Given a convex geometric shape S, a geometric graph $G_S(P)$ on P is defined to have an edge between two points if and only if there exists a homothet of S having the two points on its boundary and whose interior is empty of points of P. A matching in $G_S(P)$ is said to be *strong*, if the homothets of S representing the edges of the matching are pairwise disjoint, i.e., they do not share any point in the plane. We consider the problem of computing a strong matching in $G_S(P)$, where S is a diametral disk, an equilateral triangle, or a square. We present an algorithm that computes a strong matching in $G_S(P)$; if S is a diametral-disk, then it computes a strong matching of size at least $\lceil \frac{n-1}{17} \rceil$, and if S is an equilateral-triangle, then it computes a strong matching of size at least $\lceil \frac{n-1}{9} \rceil$. If S can be a downward or an upward equilateral-triangle, we compute a strong matching of size at least $\lceil \frac{n-1}{4} \rceil$ in $G_S(P)$. When S is an axis-aligned square, we compute a strong matching of size at least $\lceil \frac{n-1}{4} \rceil$ in $G_S(P)$, that improves the previous lower bound of $\lceil \frac{n}{5} \rceil$.

1 Introduction

Let S be a compact and convex set in the plane that contains the origin in its interior. A *homothet* of S is obtained by scaling S with respect to the origin by some factor $\mu \geq 0$, followed by a translation to a point b in the plane: $b + \mu S = \{b + \mu a : a \in S\}$. For a point set P in the plane, we define $G_S(P)$ as the geometric graph on P that has a straight-line edge between two points p and q if and only if there exists a homothet of S having p and q on its boundary and whose interior does not contain any point of P. If P is in "general position", i.e., no four points of P lie on the boundary of any homothet of S, then $G_S(P)$ is plane [9]. Hereafter, we assume that P is a set of n points in the plane that is in general position with respect to S (see Definition 1 for a formal definition). If S is a disk $\bigcirc$ whose center is the origin, then $G_{\bigcirc}(P)$ is the Delaunay triangulation of P. If S is an equilateral triangle $\bigtriangledown$ whose barycenter is the origin, then $G_{\bigtriangledown}(P)$ is the triangular-distance Delaunay graph of P, which has been introduced by Chew [10].

A *matching* in a graph G is a set of edges that do not share any vertices. A *maximum matching* is a matching of maximum cardinality. A *perfect matching* is a matching that matches all the vertices of G. Let $\mathcal{M}$ be a matching in $G_S(P)$. The matching $\mathcal{M}$ is referred to as a *matching of points with shape S*, e.g., a matching in $G_{\bigcirc}(P)$ is a matching of points with disks.

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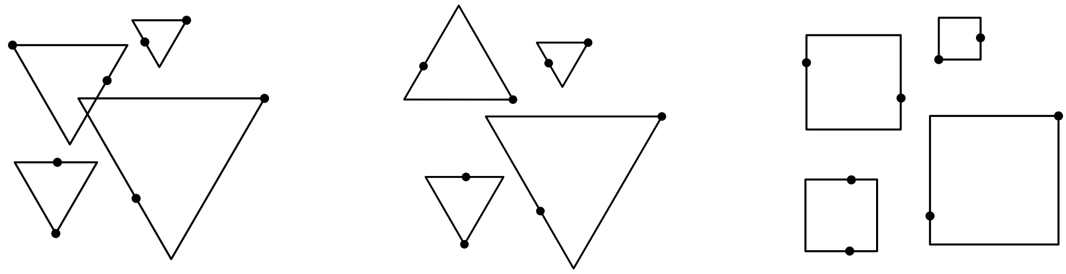
Let $\mathcal{S}_\mathcal{M}$ be a set of homothets of S representing the edges of $\mathcal{M}$. The matching $\mathcal{M}$ is called a *strong matching* if there exists a set $\mathcal{S}_\mathcal{M}$ whose elements are pairwise disjoint, i.e., the objects in $\mathcal{S}_\mathcal{M}$ do not share any point in the plane. Otherwise, $\mathcal{M}$ is a *weak matching*. See Figure 1. To be consistent with the definition of the matching in the graph theory, we use the term "matching" to refer to a weak matching. Given a point set P in the plane and a shape S, the (*strong*) *matching problem* is to compute a (strong) matching of maximum cardinality in $G_S(P)$.



(a) (b) (c)

Figure 1: Point set P and (a) a perfect weak matching in $G_{\bigtriangledown}(P)$, (b) a perfect strong matching in $G_{\davidsstar}(P)$, and (c) a perfect strong matching in $G_{\Box}(P)$.

Let $\bigcirc$ denote a closed disk whose center is the origin. Let $\Box$ denote a closed axis-aligned square whose center is the origin. Let $\bigtriangledown$ denote a closed downward equilateral triangle whose barycenter is the origin and whose lowest vertex is on the negative y-axis. For two points p and q, the closed disk that has the line segment pq as its diameter is called the diametral-disk between p and q. Let $\ominus$ denote a diametral-disk between two points.

Let P be a set of points in the plane. $G_{\bigcirc}(P)$ is the graph that has an edge between two points $p, q \in P$ if there exists a homothet of $\bigcirc$ that has p and q on its boundary and does not contain any point of P in its interior. Similarly, $G_{\Box}(P)$ is the graph that has an edge between two points $p, q \in P$ if there exists a homothet of $\Box$ that has p and q on its boundary and does not contain any point of P in its interior. $G_{\ominus}(P)$ is the graph that has an edge between two points $p, q \in P$ if the diametral-disk between p and q and does not contain any point of P in its interior. $G_{\bigtriangledown}(P)$ is the graph that has an edge between two points $p, q \in P$ if there exists a homothet of $\bigtriangledown$ that has p and q on its boundary and does not contain any point of P in its interior. If we consider an upward triangle $\triangle$, then $G_{\triangle}(P)$ is defined similarly. The graph $G_{\davidsstar}(P)$ is defined as the union of $G_{\bigtriangledown}(P)$ and $G_{\triangle}(P)$.

Definition 1. *Given a point set P and a shape $S \in \{\bigcirc, \ominus, \bigtriangledown, \Box\}$, we say that P is in "general position" with respect to S if*

$S = \bigcirc$: *no four points of P lie on the boundary of any homothet of $\bigcirc$.*

$S = \ominus$: *no four points of P lie on the boundary of any $\ominus$ between any two points of P.*

$S = \bigtriangledown$: *the line passing through any two points of P does not make angles $0°$, $60°$, or $120°$ with the horizontal. This implies that no four points of P are on the boundary of any homothet of $\bigtriangledown$.*

$S = \Box$: (*i*) *no two points in P have the same x-coordinate or the same y-coordinate, and* (*ii*) *no four points of P lie on the boundary of any homothet of $\Box$.*

In this paper we consider the strong matching problem of points in general position in the plane with respect to a given shape. Let P be a set of points in the plane that is in general

Table 1: Lower bounds on the size of weak and strong matchings in $G_S(P)$.

S	weak matching	reference	strong matching	reference
$\bigcirc$	$\lfloor \frac{n}{2} \rfloor$	[11]	$\lceil \frac{n-1}{8} \rceil$	[1]
$\ominus$	$\lceil \frac{n-1}{4} \rceil$	[7]	$\lceil \frac{n-1}{17} \rceil$	Theorem 2
$\bigtriangledown$	$\lceil \frac{n-1}{3} \rceil$	[3]	$\lceil \frac{n-1}{9} \rceil$	Theorem 3
$\bigtriangledown$ or $\bigtriangleup$	$\lceil \frac{n-1}{3} \rceil$	[3]	$\lceil \frac{n-1}{4} \rceil$	Theorem 5
$\Box$	$\lfloor \frac{n}{2} \rfloor$	[1, 2]	$\lceil \frac{n}{5} \rceil$ $\lceil \frac{n-1}{4} \rceil$	[1, 2] Theorem 4

position with respect to $S \in \{\bigcirc, \ominus, \bigtriangledown, \Box\}$. If $S = \bigcirc$, then $G_{\bigcirc}(P)$ is the Delaunay triangulation of P, $DT(P)$. If $S = \Box$, then $G_{\Box}(P)$ is the L_∞-Delaunay graph of P. If $S = \ominus$, then $G_{\ominus}(P)$ is the Gabriel graph of P, $GG(P)$. If $S = \bigtriangledown$, then $G_{\bigtriangledown}(P)$ is the half-theta six graph of P, $\frac{1}{2}\Theta_6(P)$ [8], that is in turn the triangular-distance Delaunay graph of P, which was introduced by Chew [10]. Moreover, $G_{\text{✡}}(P)$ is the theta six graph of P, $\Theta_6(P)$ [8].

1.1 Previous Work

Let P be a set of n points in the plane that is in general position with respect to a given shape $S \in \{\bigcirc, \ominus, \bigtriangledown, \Box\}$. The problem of computing a maximum (strong) matching in $G_S(P)$ is one of the fundamental problems in computational geometry and graph theory [1, 2, 3, 5, 7, 6, 11].

Dillencourt [11] and Ábrego et al. [1] considered the problem of matching points with disks. Dillencourt [11] proved that $G_{\bigcirc}(P)$ contains a perfect matching. Ábrego et al. [1] proved that $G_{\bigcirc}(P)$ has a strong matching of size at least $\lceil (n-1)/8 \rceil$. They also showed that for arbitrarily large n, there exists a set P of n points in the plane such that $G_{\bigcirc}(P)$ does not contain a strong matching of size more than $\frac{36}{73}n$. As for diametral disks, Biniaz et al. [7] proved that $G_{\ominus}(P)$ has a matching of size at least $\lceil (n-1)/4 \rceil$, and that this bound is tight.

The problem of matching of points with equilateral triangles has been considered by Babu et al. [3]. They proved that $G_{\bigtriangledown}(P)$ has a matching of size at least $\lceil (n-1)/3 \rceil$, and that this bound is tight. Since $G_{\bigtriangledown}(P)$ is a subgraph of $G_{\text{✡}}(P)$, the lower bound of $\lceil (n-1)/3 \rceil$ on the size of a maximum matching in $G_{\bigtriangledown}(P)$ holds also for $G_{\text{✡}}(P)$.

The problem of strong matching of points with axis-aligned rectangles is trivial. An obvious algorithm is to repeatedly match the two leftmost points. The problem of matching points with axis-aligned squares was considered by Ábrego et al. [1, 2]. They proved that $G_{\Box}(P)$ has a perfect matching and a strong matching of size at least $\lceil n/5 \rceil$. Further, they showed that there exists a set P of n points in the plane with arbitrarily large n, such that $G_{\Box}(P)$ does not contain a strong matching of size more than $\frac{5}{11}n$. Table 1 summarizes the results.

Bereg et al. [5] concentrated on matching points of P with axis-aligned rectangles and squares, where P is not necessarily in general position. They proved that any set of n points in the plane has a strong rectangle matching of size at least $\lfloor \frac{n}{3} \rfloor$, and that such a matching can be computed in $O(n \log n)$ time. As for squares, they presented a $\Theta(n \log n)$-time algorithm that decides whether a given matching has a weak square realization, and an $O(n^2 \log n)$-time algorithm for the strong square matching realization. They also proved that it is NP-hard to decide whether a given point set has a perfect strong square matching.

1.2 Our results

In this paper we consider the problem of computing a strong matching in $G_S(P)$, where $S \in \{\ominus, \bigtriangledown, \Box\}$. In Section 2, we provide some observations and prove necessary lemmas. Given a point set P in general position with respect to a given shape S, in Section 3, we present an algorithm that computes a strong matching in $G_S(P)$. In Section 4, we prove that if S is a diametral disk, then the algorithm of Section 3 computes a strong matching of size at least $\lceil (n-1)/17 \rceil$ in $G_{\ominus}(P)$. In Section 5, we prove that if S is an equilateral triangle, then the algorithm of Section 3 computes a strong matching of size at least $\lceil (n-1)/9 \rceil$ in $G_{\triangledown}(P)$. In Section 6, we compute a strong matching of size at least $\lceil (n-1)/4 \rceil$ in $G_{\square}(P)$; this improves the previous lower bound of $\lceil n/5 \rceil$. In Section 7, we compute a strong matching of size at least $\lceil (n-1)/4 \rceil$ in $G_{\text{✡}}(P)$. A summary of the results is given in Table 1. In Section 8 we discuss a possible way to further improve upon the result obtained for diametral-disks in Section 4. Concluding remarks and open problems are given in Section 9.

2 Preliminaries

Let $S \in \{\ominus, \triangledown\}$, and let S_1 and S_2 be two homothets of S. We say that S_1 is *smaller than* S_2 if the area of S_1 is smaller than the area of S_2. For two points $p, q \in P$, let $S(p,q)$ be a smallest homothet of S having p and q on its boundary. If S is a diametral-disk or a downward equilateral-triangle, then we denote $S(p,q)$ by $D(p,q)$ or $t(p,q)$, respectively. If S is a diametral-disk, then $D(p,q)$ is uniquely defined by p and q. If S is an equilateral-triangle, then S has the *shrinkability* property: if there exists a homothet S' of S that contains two points p and q, then there exists a homothet S'' of S such that $S'' \subseteq S'$, and p and q are on the boundary of S''. Moreover, we can shrink S'' further, such that each side of S'' contains either p or q. Then, $t(p,q)$ is uniquely defined by p and q. Thus, we have the following observation:

Observation 1. *For two points $p, q \in P$,*

- *$D(p,q)$ is uniquely defined by p and q, and it has the line segment pq as a diameter.*
- *$t(p,q)$ is uniquely defined by p and q, and it has one of p and q on a corner and the other point is on the side opposite to that corner.*

Figure 2: Illustration of Observation 2.

Given a shape $S \in \{\ominus, \triangledown\}$, we define an order on the homothets of S. Let S_1 and S_2 be two homothets of S. We say that $S_1 \prec S_2$ if the area of S_1 is less than the area of S_2. Similarly, $S_1 \preceq S_2$ if the area of S_1 is less than or equal to the area of S_2. We denote the homothet with the larger area by $\max\{S_1, S_2\}$. As illustrated in Figure 2, if $S(p,q)$ contains a point r, then both $S(p,r)$ and $S(q,r)$ have smaller area than $S(p,q)$. Thus, we have the following observation:

Observation 2. *If $S(p,q)$ contains a point r in its interior, then $\max\{S(p,r), S(q,r)\} \prec S(p,q)$.*

Given a point set P in general position with respect to a given shape $S \in \{\ominus, \triangledown\}$, let $K_S(P)$ be a complete edge-weighted geometric graph on P. For each edge $e = (p,q)$ in $K_S(P)$, we define

$S(e)$ to be the shape $S(p,q)$, i.e., a smallest homothet of S having p and q on its boundary. We say that $S(e)$ *represents* e, and vice versa. Furthermore, let the weight $w(e)$ (resp. $w(p,q)$) of e be equal to the area of $S(e)$. Thus,

$$w(p,q) < w(r,s) \quad \text{if and only if} \quad S(p,q) \prec S(r,s).$$

Note that $G_S(P)$ is a subgraph of $K_S(P)$, and has an edge (p,q) if and only if $S(p,q)$ does not contain any point of $P \setminus \{p,q\}$.

Lemma 1. *Let P be a set of n points in the plane that is in general position with respect to a given shape $S \in \{\ominus, \bigtriangledown\}$. Then, any minimum spanning tree of $K_S(P)$ is a subgraph of $G_S(P)$.*

Proof. The proof is by contradiction. Assume there exists an edge $e = (p,q)$ in a minimum spanning tree T of $K_S(P)$ such that $e \notin G_S(P)$. Since (p,q) is not an edge in $G_S(P)$, $S(p,q)$ contains a point $r \in P \setminus \{p,q\}$. By Observation 2, $\max\{S(p,r), S(q,r)\} \prec S(p,q)$. Thus, $w(p,r) < w(p,q)$ and $w(q,r) < w(p,q)$. By replacing the edge (p,q) in T with either (p,r) or (q,r), we obtain a spanning tree in $K_S(P)$ that is shorter than T. This contradicts the minimality of T. □

Lemma 2. *Let G be an edge-weighted graph with edge set E and edge-weight function $w : E \to \mathbb{R}^+$. For any cycle C in G, if the maximum-weight edge in C is unique, then that edge is not in any minimum spanning tree of G.*

Proof. The proof is by contradiction. Let $e = (u,v)$ be the unique maximum-weight edge in a cycle C in G such that e is in a minimum spanning tree T of G. Let T_u and T_v be the two trees obtained by removing e from T. Let $e' = (x,y)$ be an edge in C that connects a vertex $x \in T_u$ to a vertex $y \in T_v$. By assumption, $w(e') < w(e)$. Thus, by replacing e with e' in T, we obtain a tree $T' = T_u \cup T_v \cup \{(x,y)\}$ in G such that $w(T') < w(T)$. This contradicts the minimality of T. □

Recall that $t(p,q)$ is the smallest homothet of $\bigtriangledown$ that has p and q on its boundary. Similarly, let $t'(p,q)$ denote the smallest upward equilateral-triangle $\bigtriangleup$ having p and q on its boundary. Note that $t'(p,q)$ is uniquely defined by p and q, and it has one of p and q on a corner and the other point is on the side opposite to that corner. In addition the area of $t'(p,q)$ is equal to the area of $t(p,q)$.

Note that $G_{\bigtriangledown}(P)$ is the triangular-distance Delaunay graph *TD-DG*(P), that is in turn a half theta-six graph $\frac{1}{2}\Theta_6(P)$ [8]. A half theta-six graph on P, and equivalently $G_{\bigtriangledown}(P)$, can be constructed in the following way. For each point p in P, let l_p be the horizontal line through p. Define l_p^γ as the line obtained by rotating l_p by γ degrees in counter-clockwise direction around p. Thus, $l_p^0 = l_p$. Consider the three lines l_p^0, l_p^{60}, and l_p^{120}, which partition the plane into six disjoint cones with apex p. Let $C_p^1, \dots, C_p^6$ be the cones in counter-clockwise order around p as shown in Figure 3. C_p^1, C_p^3, C_p^5 will be referred to as *odd cones*, and C_p^2, C_p^4, C_p^6 will be referred to as *even cones*. For each even cone C_p^i, connect p to the "nearest" point q in C_p^i. The *distance* between p and q, is defined as the Euclidean distance between p and the orthogonal projection of q onto the bisector of C_p^i. See Figure 3. In other words, the nearest point to p in C_p^i is a point q in C_p^i that minimizes the area of $t(p,q)$. The resulting graph is the half theta-six graph, which is defined by even cones [8]. Moreover, the resulting

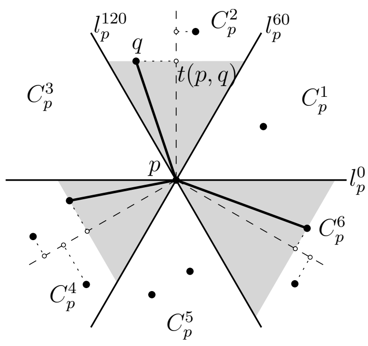


Figure 3: The construction of $G_{\bigtriangledown}(P)$.

graph is $G_{\triangledown}(P)$ that is defined with respect to the homothets of $\triangledown$. By considering the odd cones, $G_{\triangle}(P)$ is obtained. By considering the odd cones and the even cones, $G_{\text{✡}}(P)$—that is equal to $\Theta_6(P)$—is obtained. Note that $G_{\text{✡}}(P)$ is the union of $G_{\triangledown}(P)$ and $G_{\triangle}(P)$.

Let $X(p,q)$ be the regular hexagon centered at p that has q on its boundary, and its sides are parallel to l_p^0, l_p^{60}, and l_p^{120}. Then, we have the following observation:

Observation 3. *If $X(p,q)$ contains a point r in its interior, then $t(p,r) \prec t(p,q)$.*

3 Strong Matching in $G_S(P)$

Given a point set P in general position with respect to a given shape $S \in \{\ominus, \triangledown\}$, in this section we present an algorithm that computes a strong matching in $G_S(P)$. Recall that $K_S(P)$ is the complete edge-weighted graph on P with the weight of each edge e is equal to the area of $S(e)$, where $S(e)$ is a smallest homothet of S representing e. Let T be a minimum spanning tree of $K_S(P)$. By Lemma 1, T is a subgraph of $G_S(P)$. For each edge $e \in T$ we denote by $T(e^+)$ the set of all edges in T whose weight is at least $w(e)$. Moreover, we define the *influence set* of e as the set of all edges in $T(e^+)$ whose representing shapes overlap with $S(e)$, i.e.,

$$\text{Inf}(e) = \{e' \colon e' \in T(e^+), S(e') \cap S(e) \neq \emptyset\}.$$

Note that $\text{Inf}(e)$ is not empty, as $e \in \text{Inf}(e)$. Consequently, we define the *influence number* of T to be the maximum size of a set among the influence sets of edges in T, i.e.,

$$\text{Inf}(T) = \max\{|\text{Inf}(e)| \colon e \in T\}.$$

Algorithm 1 receives P and S as input and computes a strong matching in P with respect to S as follows. The algorithm starts by computing $G_S(P)$, where the weight of each edge is equal to the area of its representing shape. Then it computes a minimum spanning tree T of $G_S(P)$. Then it initializes a forest F by T, and a matching $\mathcal{M}$ by an empty set. Afterwards, as long as F is not empty, the algorithm adds the smallest edge e in F to $\mathcal{M}$, and removes the influence set of e from F. Finally, it returns $\mathcal{M}$.

Algorithm 1 *StrongMatching*(P, S)

compute $G_S(P)$
$T \leftarrow MST(G_S(P))$
$F \leftarrow T$
$\mathcal{M} \leftarrow \emptyset$
while $F \neq \emptyset$ **do**
    $e \leftarrow$ smallest edge in F
    $\mathcal{M} \leftarrow \mathcal{M} \cup \{e\}$
    $F \leftarrow F - \text{Inf}(e)$
return $\mathcal{M}$

Theorem 1. *Given a set P of n points in the plane and a shape $S \in \{\ominus, \triangledown\}$, Algorithm 1 computes a strong matching of size at least $\lceil \frac{n-1}{\text{Inf}(T)} \rceil$ in $G_S(P)$, where T is a minimum spanning tree of $G_S(P)$.*

Proof. Let $\mathcal{M}$ be the matching returned by Algorithm 1. First we show that $\mathcal{M}$ is a strong matching. If $\mathcal{M}$ contains one edge, then trivially, $\mathcal{M}$ is a strong matching. Consider any two edges e_1 and e_2 in $\mathcal{M}$. Without loss of generality assume that e_1 is considered before e_2 in the while loop. At the time e_1 is added to $\mathcal{M}$, the algorithm removes the edges in $\text{Inf}(e_1)$ from F, i.e., all the edges whose representing shapes intersect $S(e_1)$. Since e_2 remains in F after the removal of $\text{Inf}(e_1)$, we know that $e_2 \notin \text{Inf}(e_1)$. This implies that $S(e_1) \cap S(e_2) = \emptyset$, and hence $\mathcal{M}$ is a strong matching.

In each iteration of the while loop we select e as the smallest edge in F, where F is a subgraph of T. Then, all edges in F have weight at least $w(e)$. Thus, $F \subseteq T(e^+)$; that implies that the set of edges in F whose representing shapes intersect $S(e)$ is a subset of $\text{Inf}(e)$. Therefore, in each iteration of the while loop, out of at most $|\text{Inf}(e)|$ many edges of T, we add one edge to $\mathcal{M}$. Since $|\text{Inf}(e)| \leq \text{Inf}(T)$ and T has $n-1$ edges, we conclude that $|\mathcal{M}| \geq \lceil \frac{n-1}{\text{Inf}(T)} \rceil$. □

Remark Let T be the minimum spanning tree computed by Algorithm 1. Let $e = (u, v)$ be an edge in T. Recall that $T(e^+)$ contains all the edges of T whose weight is at least $w(e)$. We define the *degree* of e as $\deg(e) = \deg(u) + \deg(v) - 1$, where $\deg(u)$ and $\deg(v)$ are the numbers of edges incident to u and v in $T(e^+)$, respectively. Note that all the edges incident to u or v in $T(e^+)$ are in the influence set of e. Thus, $|\text{Inf}(e)| \geq \deg(e)$, and consequently $\text{Inf}(T) \geq \deg(e)$.

4 Strong Matching in $G_\ominus(P)$

In this section we consider the case where S is a diametral-disk $\ominus$. Recall that $G_\ominus(P)$ is an edge-weighted geometric graph, where the weight of an edge (p, q) is equal to the area of $D(p, q)$. $G_\ominus(P)$ is equal to the Gabriel graph, $GG(P)$. We prove that $G_\ominus(P)$, and consequently $GG(P)$, has a strong diametral-disk matching of size at least $\lceil \frac{n-1}{17} \rceil$.

We run Algorithm 1 on $G_\ominus(P)$ to compute a matching $\mathcal{M}$. By Theorem 1, $\mathcal{M}$ is a strong matching of size at least $\lceil \frac{n-1}{\text{Inf}(T)} \rceil$, where T is a minimum spanning tree in $G_\ominus(P)$. By Lemma 1, T is a minimum spanning tree of the complete graph $K_\ominus(P)$. Observe that T is a Euclidean minimum spanning tree for P as well. In order to prove the desired lower bound, we show that $\text{Inf}(T) \leq 17$. Since $\text{Inf}(T)$ is the maximum size of a set among the influence sets of edges in T, it suffices to show that for every edge e in T, the influence set of e contains at most 17 edges.

Lemma 3. *Let T be a minimum spanning tree of $G_\ominus(P)$, and let e be any edge in T. Then, $|\text{Inf}(e)| \leq 17$.*

We will prove this lemma in the rest of this section. Recall that, for each two points $p, q \in P$, $D(p, q)$ is the closed diametral-disk with diameter pq. Let $\mathcal{D}$ denote the set of diametral-disks representing the edges in T. Since T is a subgraph of $G_\ominus(P)$, we have the following observation:

Observation 4. *Each disk in $\mathcal{D}$ does not contain any point of P in its interior.*

Lemma 4. *For each pair D_i and D_j of disks in $\mathcal{D}$, D_i does not contain the center of D_j.*

Proof. Let (a_i, b_i) and (a_j, b_j) be the edges of T that correspond to D_i and D_j, respectively. Let c_i and c_j be the centers of D_i and D_j, respectively. Let C_i and C_j be the circles representing the boundaries of D_i and D_j, respectively. Without loss of generality assume that C_j is the bigger circle, i.e., $|a_i b_i| < |a_j b_j|$. By contradiction, suppose that C_j contains the center c_i of C_i. Let x and y denote the intersections of C_i and C_j. Let x_i (resp. x_j) be the intersection of C_i (resp. C_j) with the line through y and c_i (resp. c_j). Similarly, let y_i (resp. y_j) be the intersection of C_i (resp. C_j) with the line through x and c_i (resp. c_j).

As illustrated in Figure 4, the arcs $\widehat{x_i x}$, $\widehat{y_i y}$, $\widehat{x_j x}$, and $\widehat{y_j y}$ are the potential positions for the points a_i, b_i, a_j, and b_j, respectively. First we will show that the line segment $x_i x_j$ passes through x and $|a_i a_j| \leq |x_i x_j|$. The angles $\angle x_i x y$ and $\angle x_j x y$ are right angles, thus the line segment $x_i x_j$

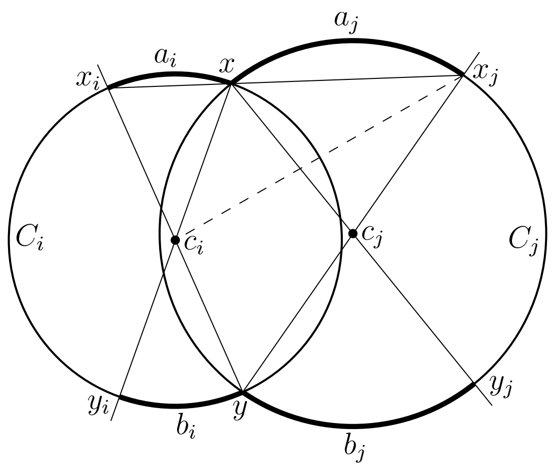


Figure 4: Illustration of Lemma 4: C_i and C_j intersect, and C_j contains the center of C_i.

goes through x. Since $\widehat{x_i x} < \pi$ (resp. $\widehat{x_j x} < \pi$), for any point $a_i \in \widehat{x_i x}, |a_i x| \leq |x_i x|$ (resp. $a_j \in \widehat{x_j x}, |a_j x| \leq |x_j x|$). Therefore,

$$|a_i a_j| \leq |a_i x| + |x a_j| \leq |x_i x| + |x x_j| = |x_i x_j|.$$

Consider triangle $\triangle x_i x_j y$, which is partitioned by segment $c_i x_j$ into $t_1 = \triangle x_i x_j c_i$ and $t_2 = \triangle c_i x_j y$. It is easy to see that $|x_i c_i|$ in t_1 is equal to $|c_i y|$ in t_2, and the segment $c_i x_j$ is shared by t_1 and t_2. Since c_i is inside C_j and $\widehat{y x_j} = \pi$, the angle $\angle y c_i x_j$ is greater than $\frac{\pi}{2}$. Thus, $\angle x_i c_i x_j$ in t_1 is smaller than $\frac{\pi}{2}$ (and hence smaller than $\angle y c_i x_j$ in t_2). That is, $|x_i x_j|$ in t_1 is smaller than $|x_j y|$ in t_2. Therefore,

$$|a_i a_j| \leq |x_i x_j| < |x_j y| = |a_j b_j|.$$

By symmetry $|b_i b_j| < |a_j b_j|$. Therefore $\max\{|a_i a_j|, |b_i b_j|\} < \max\{|a_i b_i|, |a_j b_j|\}$. Therefore, the cycle a_i, a_j, b_j, b_i, a_i contradicts Lemma 2, that is, not both (a_i, b_i) and (a_j, b_j) can be edges of T. □

Let $e = (u, v)$ be an edge in T. Without loss of generality, we suppose that $D(u, v)$ has radius 1 and is centered at the origin $o = (0, 0)$ such that $u = (-1, 0)$ and $v = (1, 0)$. For any point p in the plane, let $\|p\|$ denote the distance of p from o. Let $\mathcal{D}(e^+)$ be the disks in $\mathcal{D}$ representing the edges of $T(e^+)$. Recall that $T(e^+)$ contains the edges of T whose weight is at least $w(e)$, where $w(e)$ is equal to the area of $D(u, v)$. Since the area of any circle is directly related to its radius, we have the following observation:

Observation 5. *The disks in $\mathcal{D}(e^+)$ have radius at least* 1.

Let $C(x, r)$ (resp. $D(x, r)$) be the circle (resp. closed disk) of radius r centered at point x in the plane. Let $\mathcal{I}(e^+) = \{D_1, \ldots, D_k\}$ be the set of disks in $\mathcal{D}(e^+) \setminus \{D(u, v)\}$ intersecting $D(u, v)$. We show that $\mathcal{I}(e^+)$ contains at most sixteen disks, i.e., $k \leq 16$.

For $i \in \{1, \ldots, k\}$, let c_i denote the center of the disk D_i. In addition, let c'_i be the intersection point between $C(o, 2)$ and the ray that starts in o and passes through c_i. Let the point p_i be c_i, if $\|c_i\| < 2$, and c'_i, otherwise. See Figure 5. Finally, let $P' = \{o, u, v, p_1, \ldots, p_k\}$.

Observation 6. *Let c_j be the center of a disk D_j in $\mathcal{I}(e^+)$, where $\|c_j\| \geq 2$. Then, $D(p_j, 1) \subseteq D(c_j, \|c_j\| - 1) \subseteq D_j$. See Figure 5.*

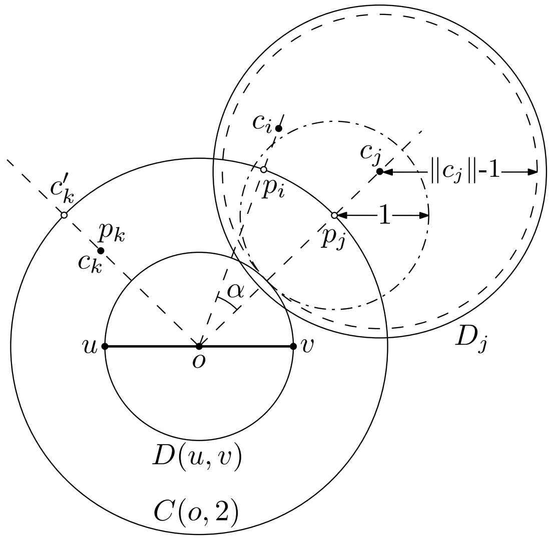


Figure 5: Proof of Lemma 5; $p_i = c'_i$, $p_j = c'_j$, and $p_k = c_k$.

Lemma 5. *The distance between any pair of points in P' is at least 1.*

Proof. Let x and y be two points in P'. We are going to prove that $|xy| \geq 1$. We distinguish between the following three cases.

- $x, y \in \{o, u, v\}$. In this case the claim is trivial.
- $x \in \{o, u, v\}, y \in \{p_1, \ldots, p_k\}$. If $\|y\| = 2$, then y is on $C(o, 2)$, and hence $|xy| \geq 1$. If $\|y\| < 2$, then y is the center of a disk D_i in $\mathcal{I}(e^+)$. By Observation 4, D_i does not contain u and v, and by Lemma 4, D_i does not contain o. Since D_i has radius at least 1, we conclude that $|xy| \geq 1$.
- $x, y \in \{p_1, \ldots, p_k\}$. Without loss of generality assume $x = p_i$ and $y = p_j$, where $1 \leq i < j \leq k$. We differentiate between three subcases:
  - $\|p_i\| < 2$ and $\|p_j\| < 2$. In this case p_i and p_j are the centers of D_i and D_j, respectively. By Lemma 4 and Observation 5, we conclude that $|p_i p_j| \geq 1$.
  - $\|p_i\| < 2$ and $\|p_j\| = 2$. By Observation 6 the disk $D(p_j, 1)$ is contained in the disk D_j. By Lemma 4, p_i is not in the interior of D_j, and consequently, it is not in the interior of $D(p_j, 1)$. Therefore, $|p_i p_j| \geq 1$.
  - $\|p_i\| = 2$ and $\|p_j\| = 2$. Recall that c_i and c_j are the centers of D_i and D_j, respectively, and that $\|c_i\| \geq 2$ and $\|c_j\| \geq 2$. Without loss of generality, assume that $\|c_i\| \leq \|c_j\|$. For the sake of contradiction assume that $|p_i p_j| < 1$. Then, for the angle $\alpha = \angle c_i o c_j$ we have $\sin(\alpha/2) < \frac{1}{4}$. Then, $\cos(\alpha) > 1 - 2\sin^2(\alpha/2) = \frac{7}{8}$. By the law of cosines in the triangle $\triangle c_i o c_j$, we have

$$|c_i c_j|^2 < \|c_i\|^2 + \|c_j\|^2 - \frac{14}{8}\|c_i\|\|c_j\|. \tag{1}$$

By Observation 6, the disk $D(c_j, \|c_j\| - 1)$ is contained in D_j; see Figure 5. By Lemma 4, c_i is not in the interior of D_j, and consequently, c_i is not in the interior of $D(c_j, \|c_j\| - 1)$. Thus, $|c_i c_j| \geq \|c_j\| - 1$. In combination with Inequality (1), this gives

$$\|c_j\|\left(\frac{14}{8}\|c_i\| - 2\right) < \|c_i\|^2 - 1. \tag{2}$$

In combination with the assumption that $\|c_i\| \leq \|c_j\|$, Inequality (2) gives

$$\frac{3}{4}\|c_i\|^2 - 2\|c_i\| + 1 < 0.$$

To satisfy this inequality, we should have $\|c_i\| < 2$, contradicting the fact that $\|c_i\| \geq 2$. This completes the proof.

□

By Lemma 5, the points in P' have mutual distance 1. Moreover, the points in P' lie in $D(o, 2)$. Bateman and Erdős [4] proved that it is impossible to have 20 points in a closed disk of radius 2 such that one of the points is at the center and all of the mutual distances are at least 1. Therefore, P' contains at most 19 points, including o, u, and v. This implies that $k \leq 16$, and hence $\mathcal{I}(e^+)$ contains at most sixteen edges. This completes the proof of Lemma 3.

Theorem 2. *Algorithm 1 computes a strong matching of size at least* $\lceil \frac{n-1}{17} \rceil$ *in* $G_\ominus(P)$.

5 Strong Matching in $G_\triangledown(P)$

In this section we consider the case where S is a downward equilateral triangle $\triangledown$ whose barycenter is the origin and one of its vertices is on the negative y-axis. In this section we assume that P is in general position, i.e., for each point $p \in P$, there is no point of $P \setminus \{p\}$ on l_p^0, l_p^{60}, and l_p^{120}. In combination with Observation 1, this implies that for two points $p, q \in P$, no point of $P \setminus \{p, q\}$ is on the boundary of $t(p, q)$ (resp. $t'(p, q)$). Recall that $t(p, q)$ is the smallest homothet of $\triangledown$ having one of p and q on a corner and the other point on the side opposite to that corner. We prove that $G_\triangledown(P)$, and consequently $\frac{1}{2}\Theta_6(P)$, has a strong triangle matching of size at least $\lceil \frac{n-1}{9} \rceil$.

We run Algorithm 1 on $G_\triangledown(P)$ to compute a matching $\mathcal{M}$. Recall that $G_\triangledown(P)$ is an edge-weighted graph where the weight of each edge (p, q) is equal to the area of $t(p, q)$. By Theorem 1, $\mathcal{M}$ is a strong matching of size at least $\lceil \frac{n-1}{\text{Inf}(T)} \rceil$, where T is a minimum spanning tree in $G_\triangledown(P)$. In order to prove the desired lower bound, we show that $\text{Inf}(T) \leq 9$. Since $\text{Inf}(T)$ is the maximum size of a set among the influence sets of edges in T, it suffices to show that for every edge e in T, the influence set of e has at most nine edges.

Lemma 6. *Let T be a minimum spanning tree of $G_\triangledown(P)$, and let e be any edge in T. Then,* $|\text{Inf}(e)| \leq 9$.

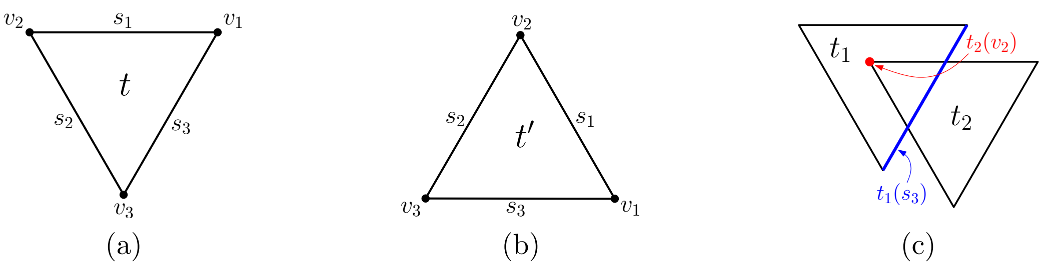


Figure 6: (a) Labeling the vertices and the sides of a downward triangle. (b) Labeling the vertices and the sides of an upward triangle. (c) Two intersecting triangles.

We will prove this lemma in the rest of this section. We label the vertices and the sides of a downward equilateral-triangle, t, and an upward equilateral-triangle, t', as depicted in

Figures 6(a) and 6(b). We refer to a vertex v_i and a side s_i of a triangle t by $t(v_i)$ and $t(s_i)$, respectively.

Recall that F is a subgraph of the minimum spanning tree T in $G_{\bigtriangledown}(P)$. In each iteration of the while loop in Algorithm 1, let $\mathcal{T}$ denote the set of triangles representing the edges in F. By Lemma 1 and the general position assumption we have

Observation 7. *Let $t(p,q)$ be a triangle in $\mathcal{T}$. Then $t(p,q)$ does not contain any point of $P \setminus \{p,q\}$ in its interior or on its boundary.*

Consider two intersecting triangles $t_1(p_1,q_1)$ and $t_2(p_2,q_2)$ in $\mathcal{T}$. By Observation 1, each side of t_1 contains either p_1 or q_1, and each side of t_2 contains either p_2 or q_2. Thus, by Observation 7, we argue that no side of t_1 is completely in the interior of t_2, and vice versa. Therefore, either exactly one vertex (corner) of t_1 is in the interior of t_2, or exactly one vertex of t_2 is in the interior of t_1. Without loss of generality assume that a corner of t_2 is in the interior of t_1, as shown in Figure 6(c). In this case we say that t_1 intersects t_2 through the vertex $t_2(v_2)$, or symmetrically, t_2 intersects t_1 through the side $t_1(s_3)$.

The following two lemmas have been proved by Biniaz et al. [6] (see Figure 7(a)):

Lemma 7 (Biniaz et al. [6])**.** *Let t_1 be a downward triangle that intersects a downward triangle t_2 through $t_2(s_1)$, and let a horizontal line ℓ intersect both t_1 and t_2. Let p_1 and q_1 be two points on $t_1(s_2)$ and $t_1(s_3)$, respectively, that are above $t_2(s_1)$. Let p_2 and q_2 be two points on $t_2(s_2)$ and $t_2(s_3)$, respectively, that are above ℓ. Then,* $\max\{t(p_1,p_2), t(q_1,q_2)\} \prec \max\{t_1,t_2\}$.

Lemma 8 (Biniaz et al. [6])**.** *For every four triangles $t_1,t_2,t_3,t_4 \in \mathcal{T}$, $t_1 \cap t_2 \cap t_3 \cap t_4 = \emptyset$.*

As a consequence of Lemma 7, we have the following corollary (see Figure 7(a)):

Corollary 1. *Let t_1,t_2,t_3 be three triangles in $\mathcal{T}$. Then t_1, t_2, and t_3 cannot make a chain configuration such that t_2 intersects t_3 through $t_3(s_1)$, and t_1 intersects both t_2 and t_3 through $t_2(s_1)$ and $t_3(s_1)$.*

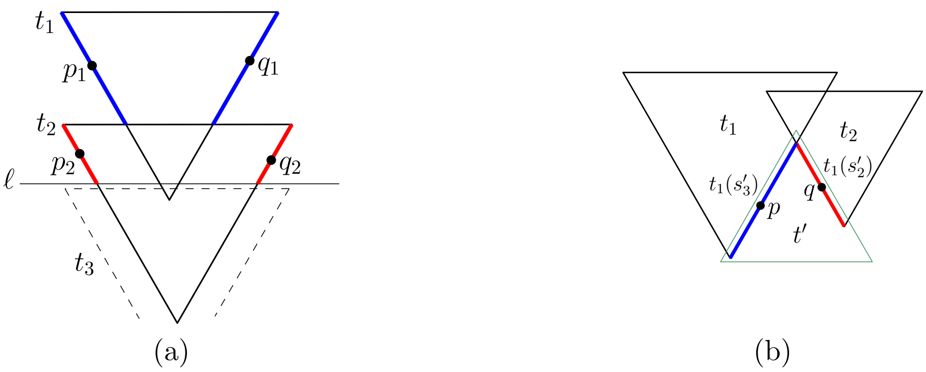


Figure 7: (a) Illustration of Lemma 7. (b) Illustration of Lemma 9.

For the following lemma refer to Figure 7(b).

Lemma 9. *Let t_1 be a downward triangle that intersects a downward triangle t_2 through $t_2(v_2)$. Let p be a point on $t_1(s_3)$ and to the left of $t_2(s_2)$, and let q be a point on $t_2(s_2)$ and to the right of $t_1(s_3)$. Then, $t(p,q) \prec \max\{t_1,t_2\}$.*

Proof. Let $t_1(s_3')$ be the part of the line segment $t_1(s_3)$ that is to the left of $t_2(s_2)$, and let $t_2(s_2')$ be the part of the line segment $t_2(s_2)$ that is to the right of $t_1(s_3)$. Without loss of generality assume that $t_1(s_3')$ is larger than $t_2(s_2')$. Let t' be an upward triangle having $t_1(s_3')$ as its left side. Then, $t' \prec t_1$, which implies that $t' \prec \max\{t_1, t_2\}$. Since t' has both p and q on its boundary, the area of the downward triangle $t(p,q)$ is smaller than the area of t'. Therefore, $t(p,q) \preceq t'$; which completes the proof. $\square$

Because of symmetry, the statement of Lemma 9 holds even if p is above $t_2(s_1)$ and q is on $t_2(s_1)$. Consider the six cones with apex at p, as shown in Figure 3.

Lemma 10. *Let T be a minimum spanning tree in $G_{\bigtriangledown}(P)$. Then, in T, every point p is adjacent to at most one point in each cone C_p^i, where $1 \leq i \leq 6$.*

Proof. If i is even, then by the construction of $G_{\bigtriangledown}(P)$, which is given in Section 2, p is adjacent to at most one point in C_p^i. So, assume that i is odd. For the sake of contradiction, assume that in T, the point p is adjacent to two points q and r in the same cone C_p^i. Then, $t(p,q)$ has q on a corner, and $t(p,r)$ has r on a corner. Without loss of generality, assume that $t(p,r) \prec t(p,q)$. Then, the hexagon $X(q,p)$ has r in its interior. Thus, $t(q,r) \prec t(p,q)$. Then the cycle r,p,q,r contradicts Lemma 2. Therefore, p is adjacent to at most one point in each of the six cones. $\square$

In Algorithm 1, in each iteration of the while loop, let $\mathcal{T}(e^+)$ be the set of triangles representing the edges of F. Recall that e is the smallest edge in F, and hence, $t(e)$ is a smallest triangle in $\mathcal{T}(e^+)$. Let $e = (p,q)$ and let $\mathcal{I}(e^+)$ be the set of triangles in $\mathcal{T}(e^+)$ (excluding $t(e)$) that intersect $t(e)$. We show that $\mathcal{I}(e^+)$ contains at most eight triangles. We partition the triangles in $\mathcal{I}(e^+)$ into $\mathcal{I}_1 \cup \mathcal{I}_2$ such that every triangle $\tau \in \mathcal{I}_1$ shares only p or q with $t = t(e) = t(p,q)$, i.e., $\mathcal{I}_1 = \{\tau : \tau \in \mathcal{I}(e^+), \tau \cap t \in \{p,q\}\}$, and every triangle $\tau \in \mathcal{I}_2$ intersects t either through a side or through a corner that is neither p nor q.

By Observation 1, for each triangle $t(p,q)$, one of p and q is a corner of $t(p,q)$ and the other one is on the side opposite to that corner. Without loss of generality, assume that p is on the corner $t(v_1)$, and hence, q is on the side $t(s_2)$. See Figure 8. Note that the other cases, where p is on $t(v_2)$ or on $t(v_3)$, are similar. Let $\tau \in \mathcal{I}_1$ represents an edge e' in T. Since the intersection of t with any triangle in $\mathcal{I}_1$ is either p or q, τ has either p or q on its boundary. In combination with Observation 7, this implies that, either p or q is an endpoint of e'. As illustrated in Figure 8, the other endpoint of e' can be either in C_p^1, C_p^2, C_p^6, or in C_q^4, because otherwise $\tau \cap t \not\subseteq \{p,q\}$. By Lemma 10, p has at most one neighbor in each of C_p^1, C_p^2, C_p^6, and q has at most one neighbor in C_q^4. Therefore, $\mathcal{I}_1$ contains at most four triangles. We are going to show that $\mathcal{I}_2$ also contains at most four triangles.

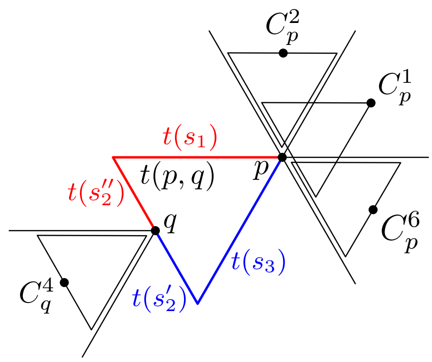


Figure 8: Illustration of the triangles in $\mathcal{I}_1$.

The point q divides $t(s_2)$ into two parts. Let $t(s_2')$ and $t(s_2'')$ be the parts of $t(s_2)$ that are below and above q, respectively; see Figure 8. The triangles in $\mathcal{I}_2$ intersect t either through $t(s_1) \cup t(s_2'')$ or through $t(s_3) \cup t(s_2')$; the two sets are shown by red and blue polylines in Figure 8. We show that at most two triangles in $\mathcal{I}_2$ intersect t through each of $t(s_1) \cup t(s_2'')$ or $t(s_3) \cup t(s_2')$. Because of symmetry, we only prove this for $t(s_3) \cup t(s_2')$. When a triangle t' intersects t through both $t(s_3)$ and $t(s_2')$, we say t' intersects t through $t(v_3)$. In the next lemma, we prove that at most one triangle in $\mathcal{I}_2$ intersects t through each of $t(s_3)$, $t(s_2')$. Again, because of symmetry, we only prove this for $t(s_3)$.

Lemma 11. *At most one triangle in $\mathcal{I}_2$ intersects t through $t(s_3)$.*

Proof. The proof is by contradiction. Assume that two triangles $t_1(p_1,q_1)$ and $t_2(p_2,q_2)$ in $\mathcal{I}_2$ intersect t through $t(s_3)$. Without loss of generality, assume that p_i is on $t_i(s_1)$ and q_i is on

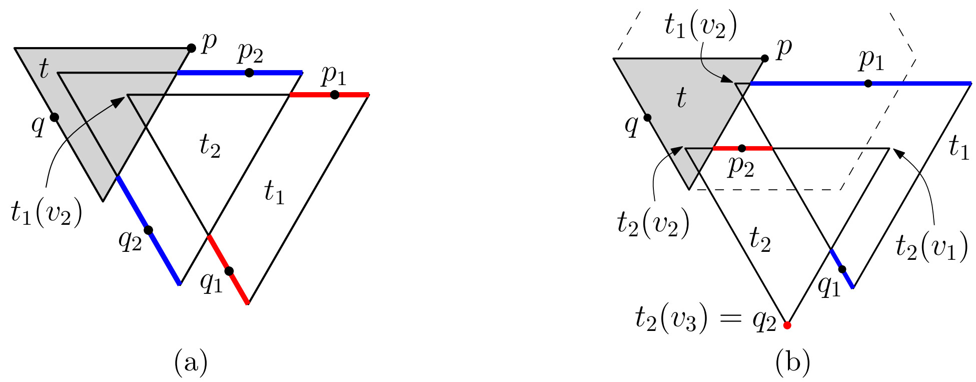


Figure 9: Illustration of Lemma 11: (a) $t_1(v_2) \in t_2$. (b) $t_1(v_2) \notin t_2$ and $t_2(v_2) \notin t_1$.

$t_i(s_2)$ for $i = 1, 2$. Recall that $t \preceq t_1$ and $t \preceq t_2$. If $t_1(v_2)$ is in the interior of t_2 (as shown in Figure 9(a)) or $t_2(v_2)$ is in the interior of t_1, then we get a contradiction to Corollary 1. Thus, we assume that $t_1(v_2) \notin t_2$ and $t_2(v_2) \notin t_1$.

Without loss of generality, assume that $t_1(s_1)$ is above $t_2(s_1)$; see Figure 9(b). By Lemma 9, we have $t(p, p_1) \prec \max\{t, t_1\} \preceq t_1$. If q_1 is in $X(p, q)$, then by Observation 3, $t(p, q_1) \prec t$. Then, the cycle p, p_1, q_1, p contradicts Lemma 2. Thus, assume that $q_1 \notin X(p, q,)$. In this case $t_2(s_3)$ is to the left of $t_1(s_3)$, because otherwise q_1 lies in t_2 which contradicts Observation 7. Since both t_1 and t_2 are larger than t, t_2 intersects t_1 through $t_1(s_2)$, and hence $t_2(v_1)$ is in the interior of t_1. This implies that $q_2 = t_2(v_3)$. In addition, p_2 is on the part of $t_2(s_1)$ that lies in the interior of $X(p, q)$. By Observation 3 and Lemma 9, we have $t(p, p_2) \prec t$ and $t(q_1, q_2) \prec \max\{t_1, t_2\}$, respectively. Thus, the cycle p, p_1, q_1, q_2, p_2, p contradicts Lemma 2. □

Lemma 12. *At most two triangles in $\mathcal{I}_2$ intersect t through $t(v_3)$.*

Proof. For the sake of contradiction assume three triangles $t_1, t_2, t_3 \in \mathcal{I}_2$ intersect t through $t(v_3)$. This implies that $t(v_3)$ belongs to four triangles t, t_1, t_2, t_3, which contradicts Lemma 8. □

Lemma 13. *If two triangles in $\mathcal{I}_2$ intersect t through $t(v_3)$, then no other triangle in $\mathcal{I}_2$ intersects t through $t(s_3)$ or through $t(s_2')$.*

Proof. The proof is by contradiction. Assume that two triangles $t_1(p_1, q_1)$ and $t_2(p_2, q_2)$ in $\mathcal{I}_2$ intersect t through $t(v_3)$, and a triangle $t_3(p_3, q_3)$ in $\mathcal{I}_2$ intersects t through $t(s_3)$ or $t(s_2')$. Let p_i be the input point that lies on $t_i(s_1)$ for $i = 1, 2, 3$. By Lemma 12, t_3 cannot intersect both $t(s_3)$ and $t(s_2')$. Thus, t_3 intersects t either through $t(s_3)$ or through $t(s_2')$. We prove the former case; the proof for the latter case is similar. Assume that t_3 intersects t through $t(s_3)$. By Lemma 9, $t(p, p_3) \prec t_3$. See Figure 10. In addition, both $t_1(s_3)$ and $t_2(s_3)$ are to the left of $t_3(s_3)$, because otherwise q_3 lies in $t_1 \cup t_2 \cup X(p, q)$. If $q_3 \in t_1 \cup t_2$, we get a contradiction to Observation 7. If $q_3 \in X(p, q)$ then by Observation 3, we have $t(p, q_3) \prec t$, and hence, the cycle p, p_3, q_3, p contradicts Lemma 2.

Without loss of generality, assume that $t_1(s_1)$ is above $t_2(s_1)$; see Figure 10. If $t_1(v_3) \in t_2$ or $t_2(v_3) \in t_1$, then we get a contradiction to Corollary 1. Thus, assume that $t_1(v_3) \notin t_2$ and $t_2(v_3) \notin t_1$. This implies that either (i) $t_2(s_3)$ is to the right of $t_1(s_3)$ or (ii) $t_2(s_2)$ is to the left of $t_1(s_2)$. We show that both cases lead to a contradiction.

In case (i), p_2 lies in the interior of $X(p, q,)$, and then by Observation 3, we have $t(p, p_2) \prec t$; see Figure 10(a). In addition, Lemma 9 implies that $t(p_2, q_3) \prec \max\{t, t_3\} \preceq t_3$. Thus, the cycle p, p_3, q_3, p_2, p contradicts Lemma 2.

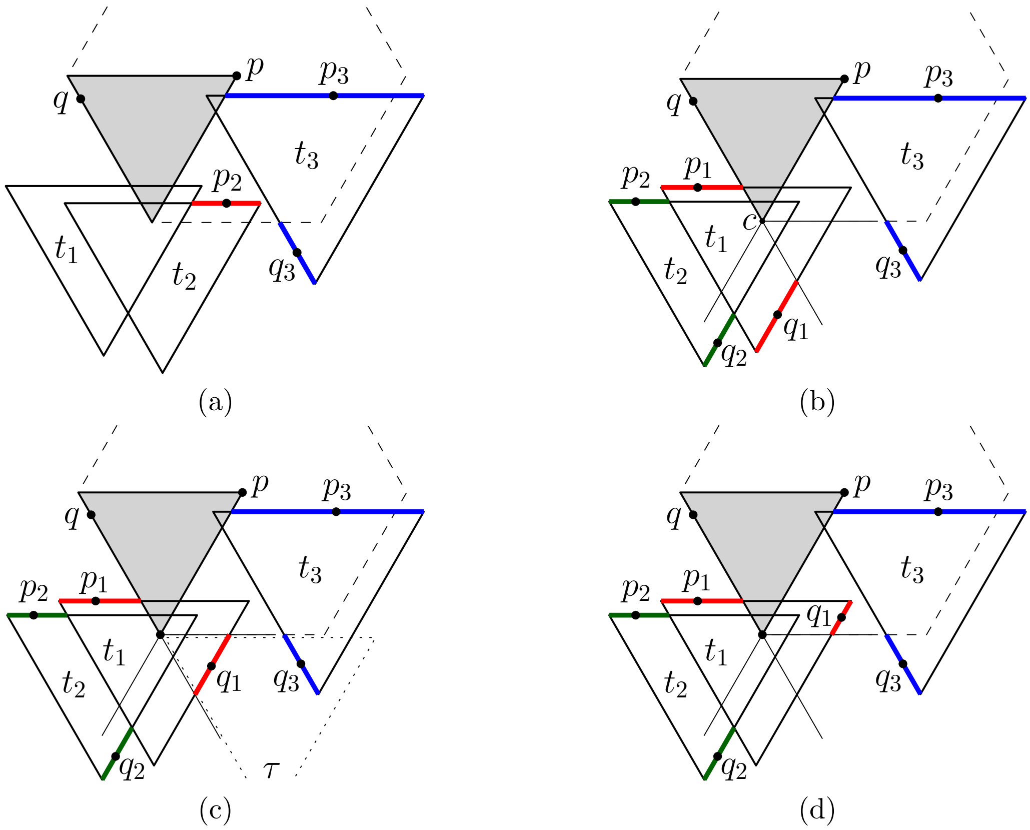


Figure 10: Illustration for the proof of Lemma 13: (a) p_2 is to the right of $t_1(s_3)$, (b) $q_1 \in C^5_{t(v_3)}$, (c) $q_1 \in C^6_{t(v_3)}$, and (d) $q_1 \in C^1_{t(v_3)}$.

Now consider case (ii) where $t_1(s_1)$ is above $t_2(s_1)$ and $t_2(s_2)$ is to the left of $t_1(s_2)$. If p_1 is to the right of t, then as in case (i), the cycle p, p_3, q_3, p_1, p contradicts Lemma 2. Thus, assume that p_1 is to the left of t, as shown in Figure 10(b). By Lemma 9, we have $t(q, p_1) \prec \max\{t, t_1\} \preceq t_1$. Each side of t_1 contains either p_1 or q_1, while p_1 is on the part of $t_1(s_1)$ that is to the left of t, thus, q_1 is on $t_1(s_3)$. Consider the six cones around $t(v_3)$; see Figure 10(b). We have three cases: (a) $q_1 \in C^5_{t(v_3)}$, (b) $q_1 \in C^6_{t(v_3)}$ or (c) $q_1 \in C^1_{t(v_3)}$.

In case (a), which is shown in Figure 10(b), by Lemma 7, we have $\max\{t(p_1, p_2), t(q_1, q_2)\} \prec \max\{t_1, t_2\}$. Thus, the cycle p_1, p_2, q_2, q_1, p_1 contradicts Lemma 2. In Case (b), which is shown in Figure 10(c), we have $t(q_1, q_3) \prec t_3$, because if we map t_3 to a downward triangle τ—of area equal to the area of t_3—that has $\tau(v_2)$ on $t(v_3)$, then τ contains both q_1 and q_3. Therefore, the cycle $p, p_3, q_3, q_1, p_1, q, p$ contradicts Lemma 2. In Case (c), which is shown in Figure 10(d), by Observation 3, $t(p, q_1) \prec t$, and then, the cycle p, q_1, p_1, q, p contradicts Lemma 2. $\square$

Lemma 14. *If three triangles intersect t through $t(s'_2), t(v_3)$ and $t(s_3)$, then at least one of the three triangles is not in $\mathcal{I}_2$.*

Proof. The proof is by contradiction. Assume that three triangles $t_1(p_1, q_1), t_2(p_2, q_2), t_3(p_3, q_3)$ in $\mathcal{I}_2$ intersect t through $t(s'_2), t(v_3), t(s_3)$, respectively. Let p_i be the point that lies on $t_i(s_1)$ for $i = 1, 2, 3$. See Figure 11(a). By Lemma 9, we have $t(p, p_3) \prec t_3$ and $t(q, p_1) \prec t_1$. If q_3 is in the interior of $X(p, q)$, then by Observation 3, $t(p, q_3) \prec t$, and hence, the cycle p, p_3, q_3, p contradicts Lemma 2. If q_1 is in $X(q, p)$, then by Observation 3, $t(q, q_1) \prec t$, and hence, the cycle q, q_1, p_1, q contradicts Lemma 2; see Figure 11(b). Thus, assume that $q_3 \notin X(p, q)$ and $q_1 \notin X(q, p)$. Let

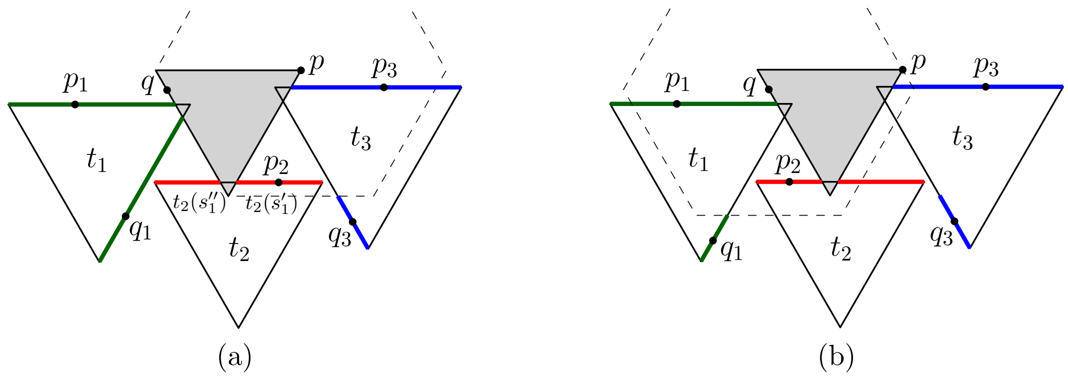


Figure 11: Illustration for the proof of Lemma 14: (a) $p_2 \in t_2(s_1')$, and (b) $p_2 \in t_2(s_1'')$.

$t_2(s_1')$ and $t_2(s_1'')$ be the parts of $t_2(s_1)$ that are to the right of $t(s_3)$ and to the left of $t(s_2)$, respectively. Consider the point p_2 that lies on $t_2(s_1)$. If $p_2 \in t_2(s_1')$, then $p_2 \in X(p,q)$ and by Observation 3, $t(p,p_2) \prec t$. In addition, Lemma 9 implies that $t(p_2,q_3) \prec t_3$. Thus, the cycle p,p_3,q_3,p_2,p contradicts Lemma 2; see Figure 11(a). If $p_2 \in t_2(s_1'')$, then $p_2 \in X(q,p)$ and by Observation 3, $t(q,p_2) \prec t$. In addition, Lemma 9 implies that $t(p_2,q_1) \prec t_2$. Thus, the cycle q,p_2,q_1,p_1,q contradicts Lemma 2; see Figure 11(b). □

Putting Lemmas 11, 12, 13, and 14 together, implies that at most two triangles in $\mathcal{I}_2$ intersect t through $t(s_3) \cup t(s_2')$, and consequently, at most two triangles in $\mathcal{I}_2$ intersect t through $t(s_1) \cup t(s_2'')$. Thus, $\mathcal{I}_2$ contains at most four triangles. Recall that $\mathcal{I}_1$ contains at most four triangles. Then, $\mathcal{I}(e^+)$ contains at most eight triangles. Therefore, the influence set of e contains at most 9 edges (including e itself). This completes the proof of Lemma 6.

Theorem 3. *Algorithm 1 computes a strong matching of size at least* $\lceil \frac{n-1}{9} \rceil$ *in* $G_{\bigtriangledown}(P)$.

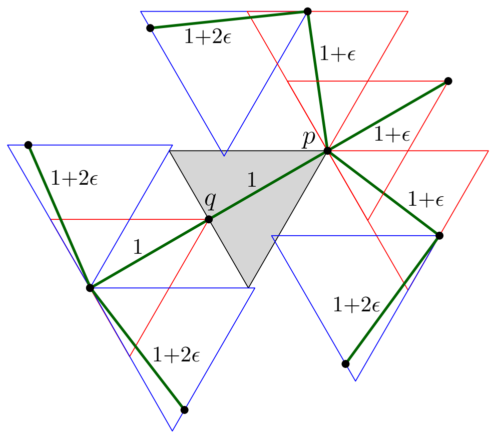


Figure 12: Four triangles in $\mathcal{I}_1$ (in red) and four triangles in $\mathcal{I}_2$ (in blue) intersect with $t(p,q)$.

The bound obtained by Lemma 6 is tight. Figure 12 shows a configuration of 10 points in general position such that the influence set of a minimal edge is 9. In Figure 12, $t = t(p,q)$ represents a smallest edge of weight 1; the minimum spanning tree is shown in bold-green line segments. The weight of all edges—the area of the triangles representing these edges—is at least 1. The red triangles are in $\mathcal{I}_1$ and share either p or q with t. The blue triangles are in $\mathcal{I}_2$ and intersect t through $t(s_1) \cup t(s_2'')$ or through $t(s_3) \cup t(s_2')$; as shown in Figure 12, two of them share only the points $t(v_2)$ and $t(v_3)$.

6 Strong Matching in $G_\square(P)$

In this section we consider the problem of computing a strong matching in $G_\square(P)$, where $\square$ is an axis-aligned square whose center is the origin. We assume that P is in general position, i.e., (i) no two points have the same x-coordinate or the same y-coordinate, and (ii) no four points are on the boundary of any homothet of $\square$. Recall that $G_\square(P)$ is equal to the L_∞-Delaunay graph on P. Ábrego et al. [1, 2] proved that $G_\square(P)$ has a strong matching of size at least $\lceil n/5 \rceil$. Using a similar approach as Ábrego et al. [1, 2], we prove that $G_\square(P)$ has a strong matching of size at least $\lceil \frac{n-1}{4} \rceil$.

Theorem 4. *Let P be a set of n points in general position in the plane. Let S be an axis-parallel square that contains P. Then, it is possible to find a strong matching of size at least $\lceil \frac{n-1}{4} \rceil$ for $G_\square(P)$ such that for each edge e in this matching, the square corresponding to e is in S.*

Proof. The proof is by induction. Assume that any point set of size $n' \leq n-1$ in an axis-parallel square S' has a strong matching of size at least $\lceil \frac{n'-1}{4} \rceil$ in S'. If n is 0 or 1, then there is no matching in S, and if $n \in \{2,3,4,5\}$, then by shrinking S, it is possible to find a strongly matched pair. Now suppose that $n \geq 6$, and $n = 4m + r$, where $r \in \{0,1,2,3\}$. If $r \in \{0,1,3\}$, then $\lceil \frac{n-1}{4} \rceil = \lceil \frac{(n-1)-1}{4} \rceil$, and by induction we are done. So we may assume that that $n = 4m+2$, for some $m \geq 1$. We prove that there are $\lceil \frac{n-1}{4} \rceil = m+1$ disjoint squares in S, each of them matching a pair of points in P. To this end we partition S into four equal area squares S_1, S_2, S_3, S_4 that contain n_1, n_2, n_3, n_4 points, respectively; see Figure 13(a). Let $n_i = 4m_i + r_i$ for $1 \leq i \leq 4$, where $r_i \in \{0,1,2,3\}$. Let R be the multiset $\{r_1, r_2, r_3, r_4\}$. By induction, in $S_1 \cup S_2 \cup S_3 \cup S_4$, we have a strong matching of size at least

$$A = \left\lceil \frac{n_1 - 1}{4} \right\rceil + \left\lceil \frac{n_2 - 1}{4} \right\rceil + \left\lceil \frac{n_3 - 1}{4} \right\rceil + \left\lceil \frac{n_4 - 1}{4} \right\rceil. \tag{3}$$

Claim 1: $A \geq m$.
Proof. By Equation (3), we have

$$A = \sum_{i=1}^{4} \left\lceil \frac{n_i - 1}{4} \right\rceil \geq \sum_{i=1}^{4} \frac{n_i - 1}{4} = \frac{n}{4} - 1 = \frac{4m+2}{4} - 1 = m - \frac{1}{2}.$$

Since A and m are integers, we argue that $A \geq m$. $\square$

If $A > m$, then we are done. Assume that $A = m$; in fact, by the induction hypothesis we have a strong matching of size at least m for P. In order to complete the proof, we have to get one more strongly matched pair. Let R be the multiset $\{r_1, r_2, r_3, r_4\}$.

Claim 2: *If $A = m$, then either (i) $R = \{1,1,1,3\}$ or (ii) $R = \{0,0,1,1\}$.*
Proof. Let $\alpha = r_1 + r_2 + r_3 + r_4$, where $0 \leq r_i \leq 3$. Then $n = 4(m_1 + m_2 + m_3 + m_4) + \alpha$. Since $n = 4m+2$, $\alpha = 4k+2$, for some $0 \leq k \leq 2$. Thus, $n = 4m+2$, where $m = m_1+m_2+m_3+m_4+k$.

By induction, in S_i, we get a matching of size at least $\lceil \frac{(4m_i+r_i)-1}{4} \rceil = m_i + \lceil \frac{r_i - 1}{4} \rceil$. Hence, in $S_1 \cup S_2 \cup S_3 \cup S_4$, we get a matching of size at least

$$A = m_1 + m_2 + m_3 + m_4 + \left\lceil \frac{r_1 - 1}{4} \right\rceil + \left\lceil \frac{r_2 - 1}{4} \right\rceil + \left\lceil \frac{r_3 - 1}{4} \right\rceil + \left\lceil \frac{r_4 - 1}{4} \right\rceil.$$

Since $A = m$ and $m = m_1 + m_2 + m_3 + m_4 + k$, we have

$$k = \left\lceil \frac{r_1 - 1}{4} \right\rceil + \left\lceil \frac{r_2 - 1}{4} \right\rceil + \left\lceil \frac{r_3 - 1}{4} \right\rceil + \left\lceil \frac{r_4 - 1}{4} \right\rceil. \tag{4}$$

Note that $0 \le k \le 2$. We go through some case analysis: (i) $k = 0$, (ii) $k = 1$, (iii) $k = 2$. In case (i), we have $\alpha = 4k+2 = r_1+r_2+r_3+r_4 = 2$. In order to have k equal to 0 in Equation (4), no element in R can be greater than 1; this happens only if two elements in R are equal to 0 and the other two elements are equal to 1. In case (ii), we have $\alpha = r_1 + r_2 + r_3 + r_4 = 6$. In order to have k equal to 1 in Equation (4), at most one element in R should be greater than 1; this happens only if three elements in R are equal to 1 and the remaining element is equal to 3 (note that all elements in R are less than 4). In case (iii), we have $\alpha = r_1 + r_2 + r_3 + r_4 = 10$. In order to have k equal to 2 in Equation (4), at most two elements in R should be greater than 1; which is not possible. $\square$

In both cases of Claim 2 we show how to augment a strong matching of size m by one more pair such that the resulting matching is strong and has size $m + 1$.

We define S_1^{-x} as the smallest axis-parallel square contained in S_1 and anchored at the top-left corner of S_1, that contains all the points in S_1 except x points. If S_1 contains less than x points, then the area of S_1^{-x} is zero. We also define S_1^{+x} as the smallest axis-parallel square that contains S_1 and anchored at the top-left corner of S_1, that has all the points in S_1 plus x other points of P. See Figure 13(a). Similarly we define the squares S_2^{-x}, S_2^{+x}, S_3^{-x}, S_3^{+x}, and S_4^{-x}, S_4^{+x} that are anchored at the top-right corner of S_2, the bottom-left corner of S_3, and the bottom-right corner of S_4, respectively.

Case (i): $R = \{1, 1, 1, 3\}$.

In this case, we have $m = m_1 + m_2 + m_3 + m_4 + 1$. Without loss of generality, assume that $r_1 = 3$ and $r_2 = r_3 = r_4 = 1$. Consider the squares S_1^{-1}, S_2^{-3}, S_3^{-3}, and S_4^{-3}. Note that the area of some of these squares—but not all—may be equal to zero. See Figure 13(b). By induction, we get matchings of sizes at least $m_1 + 1$, m_2, m_3, and m_4, in S_1^{-1}, S_2^{-3}, S_3^{-3}, and S_4^{-3}, respectively. Now consider the largest square among S_1^{-1}, S_2^{-3}, S_3^{-3}, and S_4^{-3}. Because of symmetry, we have only three cases: (i) S_1^{-1} is the largest, (ii) S_2^{-3} is the largest, and (iii) S_4^{-3} is the largest.

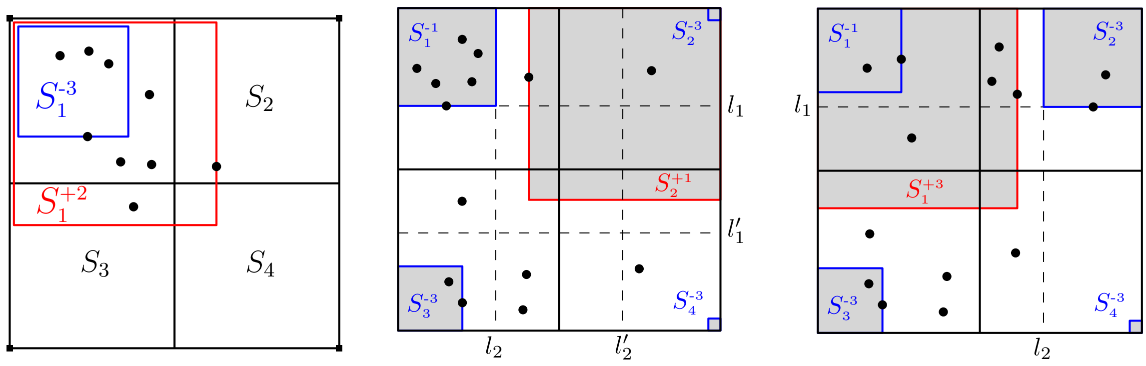


Figure 13: (a) Split S into four equal area squares. (b) S_1^{-1} is larger than S_2^{-3}, S_3^{-3}, and S_4^{-3}. (c) S_2^{-3} is larger than S_1^{-1}, S_3^{-3}, and S_4^{-3}.

- *S_1^{-1} is the largest square.* Consider the lines l_1 and l_2 that contain the bottom side and right side of S_1^{-1}, respectively; see the dashed lines in Figure 13(b). Note that l_1 and l_2 and their mirrored versions l_1' and l_2' do not intersect any of S_2^{-3}, S_3^{-3}, and S_4^{-3}. If any point of S_1 is to the right of l_2, then by induction, we get a matching of size at least $(m_1 + 1) + (m_2 + 1) + m_3 + m_4$ in $S_1^{-1} \cup S_2^{+1} \cup S_3^{-3} \cup S_4^{-3}$. Note that S_2^{+1} is separated from S_3^{-3} by l_2 and from S_4^{-3} by l_1' (since we assume that S_1^{-1} is the largest of the four squares).

Otherwise, by induction, we get a matching of size at least $(m_1+1)+m_2+(m_3+1)+m_4$ in $S_1^{-1} \cup S_2^{-3} \cup S_3^{+1} \cup S_4^{-3}$, which, again, is a disjoint union. In both cases we get a matching of size at least $m+1$ in S.

- S_2^{-3} *is the largest square.* Consider the lines l_1 and l_2 that contain the bottom side and left side of S_2^{-3}, respectively; the dashed lines in Figure 13(c). Note that l_1 and l_2 do not intersect any of S_1^{-1}, S_3^{-3}, and S_4^{-3}. If any point of S_2 is below l_1, then by induction, we get a matching of size at least $(m_1+1)+m_2+m_3+(m_4+1)$ in $S_1^{-1} \cup S_2^{-3} \cup S_3^{-3} \cup S_4^{+1}$. Otherwise, by induction, we get a matching of size at least $(m_1+2)+m_2+m_3+m_4$ in $S_1^{+3} \cup S_2^{-3} \cup S_3^{-3} \cup S_4^{-3}$; see Figure 13(c). In all cases we get a matching of size at least $m+1$ in S.

- S_4^{-3} *is the largest square.* Consider the lines l_1 and l_2 that contain the top side and left side of S_4^{-3}, respectively. If any point of S_4 is above l_1, then by induction, we get a matching of size at least $(m_1+1)+(m_2+1)+m_3+m_4$ in $S_1^{-1} \cup S_2^{+1} \cup S_3^{-3} \cup S_4^{-3}$. Otherwise, by induction, we get a matching of size at least $(m_1+1)+m_2+(m_3+1)+m_4$ in $S_1^{-1} \cup S_2^{-3} \cup S_3^{+1} \cup S_4^{-3}$. In all cases we get a matching of size at least $m+1$ in S.

Case (ii): $R = \{0, 0, 1, 1\}$.

In this case, we have $m = m_1 + m_2 + m_3 + m_4$. Due to symmetry, only the following two cases may arise:

- $r_1 = r_2 = 1$ *and* $r_3 = r_4 = 0$. Consider the squares S_1^{-3}, S_2^{-3}, S_3^{-2}, and S_4^{-2}. By induction, we get matchings of sizes at least m_1, m_2, m_3, and m_4, in S_1^{-3}, S_2^{-3}, S_3^{-2}, and S_4^{-2}, respectively. Now consider the largest square among S_1^{-3}, S_2^{-3}, S_3^{-2}, and S_4^{-2}. Because of symmetry, we have only two cases: (a) S_1^{-3} is the largest, (b) S_3^{-2} is the largest. In case (a) we get one more matched pair either in S_2^{+1} or in S_3^{+2}. In case (b) we get one more matched pair either in S_1^{+1} or in S_4^{+2}.

- $r_1 = r_4 = 1$ *and* $r_2 = r_3 = 0$. Consider the squares S_1^{-3}, S_2^{-2}, S_3^{-2}, and S_4^{-3}. By induction, we get matchings of sizes at least m_1, m_2, m_3, and m_4, in S_1^{-3}, S_2^{-2}, S_3^{-2}, and S_4^{-3}, respectively. Now consider the largest square among S_1^{-3}, S_2^{-2}, S_3^{-2}, and S_4^{-3}. Because of symmetry, we have only two cases: (a) S_1^{-3} is the largest, (b) S_2^{-2} is the largest. In case (a) we get one more matched pair either in S_2^{+2} or in S_3^{+2}. In case (b) we get one more matched pair either in S_1^{+1} or in S_4^{+1}.

□

7 Strong Matching in $G_{✡}(P)$

In this section we consider the problem of computing a strong matching in $G_{✡}(P)$. Recall that $G_{✡}(P)$ is the union of $G_{\bigtriangledown}(P)$ and $G_{\bigtriangleup}(P)$, and is equal to the graph $\Theta_6(P)$. We assume that P is in general position, i.e., for each point $p \in P$, there is no point of $P \setminus \{p\}$ on l_p^0, l_p^{60}, and l_p^{120}. A matching $\mathcal{M}$ in $G_{✡}(P)$ is a strong matching if for each edge e in $\mathcal{M}$ there is a homothet of $\bigtriangledown$ or a homothet of $\bigtriangleup$ representing e such that these homothets are pairwise disjoint. See Figure 1(b). Using a similar approach as in Section 6, we prove the following theorem:

Theorem 5. *Let P be a set of n points in general position in the plane. Let S be an upward or a downward equilateral-triangle that contains P. Then, it is possible to find a strong matching of size at least $\lceil \frac{n-1}{4} \rceil$ for $G_{✡}(P)$ such that for each edge e in this matching, the triangle corresponding to e is in S.*

Proof. The proof is by induction. Assume that any point set of size $n' \leq n-1$ in a triangle S' has a strong matching of size at least $\lceil \frac{n'-1}{4} \rceil$ in S'. Without loss of generality, assume that S is an upward equilateral-triangle. If n is 0 or 1, then there is no matching in S, and if $n \in \{2, 3, 4, 5\}$, then by shrinking S, it is possible to find a strongly matched pair; the statement of the theorem holds. Now suppose that $n \geq 6$, and $n = 4m + r$, where $r \in \{0, 1, 2, 3\}$. If $r \in \{0, 1, 3\}$, then $\lceil \frac{n-1}{4} \rceil = \lceil \frac{(n-1)-1}{4} \rceil$, and by induction we are done. So we may assume that $n = 4m + 2$, for some $m \geq 1$. We prove that there are $\lceil \frac{n-1}{4} \rceil = m + 1$ disjoint equilateral-triangles (upward or downward) in S, each of them matching a pair of points in P. To this end we partition S into four equal area equilateral triangles S_1, S_2, S_3, S_4 containing n_1, n_2, n_3, n_4 points, respectively; see Figure 14(a). Let $n_i = 4m_i + r_i$, where $r_i \in \{0, 1, 2, 3\}$. By induction, in $S_1 \cup S_2 \cup S_3 \cup S_4$, we have a strong matching of size at least

$$A = \left\lceil \frac{n_1 - 1}{4} \right\rceil + \left\lceil \frac{n_2 - 1}{4} \right\rceil + \left\lceil \frac{n_3 - 1}{4} \right\rceil + \left\lceil \frac{n_4 - 1}{4} \right\rceil .$$

In the proof of Theorem 4, we have shown the following two claims:

Claim 1: *$A \geq m$.*

Claim 2: *If $A = m$, then either (i) $R = \{1, 1, 1, 3\}$ or (ii) $R = \{0, 0, 1, 1\}$.*

If $A > m$, then we are done. Assume that $A = m$; in fact, by the induction hypothesis we have an strong matching of size at least m in S. By Claim 2 we have two cases. In both cases of Claim 2 we show how to augment a strong matching of size m by one more pair such that the resulting matching is strong and has size $m + 1$. We show how to find one more strongly matched pair in each case of Claim 2.

We define S_1^{-x} as the smallest upward equilateral-triangle contained in S_1 and anchored at the top corner of S_1, that contains all the points in S_1 except x points. If S_1 contains less than x points, then the area of S_1^{-x} is zero. We also define S_1^{+x} as the smallest upward equilateral-triangle that contains S_1 and anchored at the top corner of S_1, that has all the points in S_1 plus x other points of P. Similarly we define upward triangles S_2^{-x} and S_2^{+x} that are anchored at the left corner of S_2. Moreover, we define upward triangles S_4^{-x} and S_4^{+x} that are anchored at the right corner of S_4. We define downward triangles $S_{3\mathrm{l}}^{-x}$, $S_{3\mathrm{r}}^{-x}$, $S_{3\mathrm{b}}^{-x}$ that are anchored at the top-left corner, top-right corner, and bottom corner of S_3, respectively. See Figure 14(a).

Case (i): $R = \{1, 1, 1, 3\}$.

In this case, we have $m = m_1 + m_2 + m_3 + m_4 + 1$. Because of symmetry, we have two cases: (i) $r_3 = 3$, (ii) $r_j = 3$ for some $j \in \{1, 2, 4\}$.

- $r_3 = 3$.

  In this case $n_3 = 4m_3 + 3$. We differentiate between two cases: the case that all the elements of the multiset $\{m_1, m_2, m_4\}$ are equal to zero, and the case that some of them are greater than zero.

  - *All elements of $\{m_1, m_2, m_4\}$ are equal zero.* In this case, we have $m = m_3 + 1$. Consider the triangles S_2^{+1} and $S_{3\mathrm{r}}^{-1}$. See Figure 14(a). Note that S_2^{+1} and $S_{3\mathrm{r}}^{-1}$ are disjoint, S_2^{+1} contains two points, and $S_{3\mathrm{r}}^{-1}$ contains $4m_3 + 2$ points. By induction, we get a matched pair in S_2^{+1} and a matching of size at least $m_3 + 1$ in $S_{3\mathrm{r}}^{-1}$. Thus, in total, we get a matching of size at least $1 + (m_3 + 1) = m + 1$ in S.

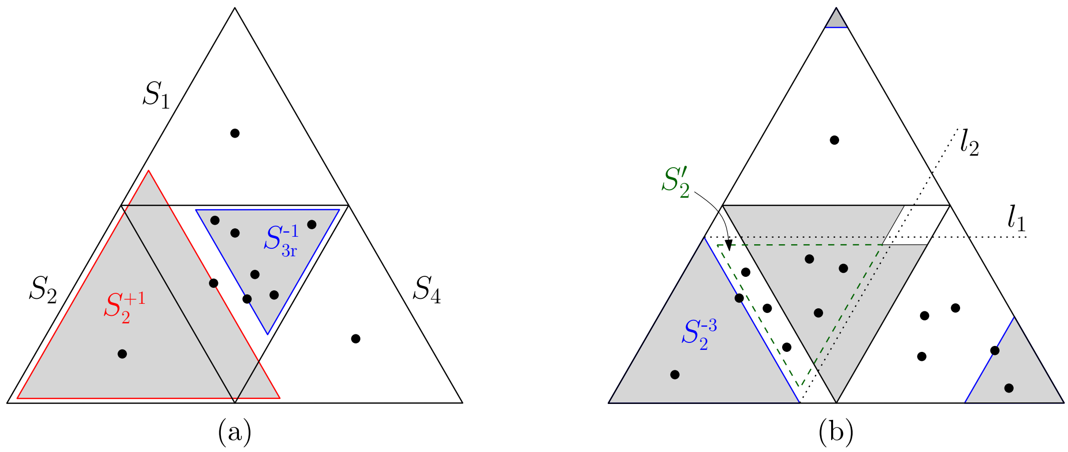


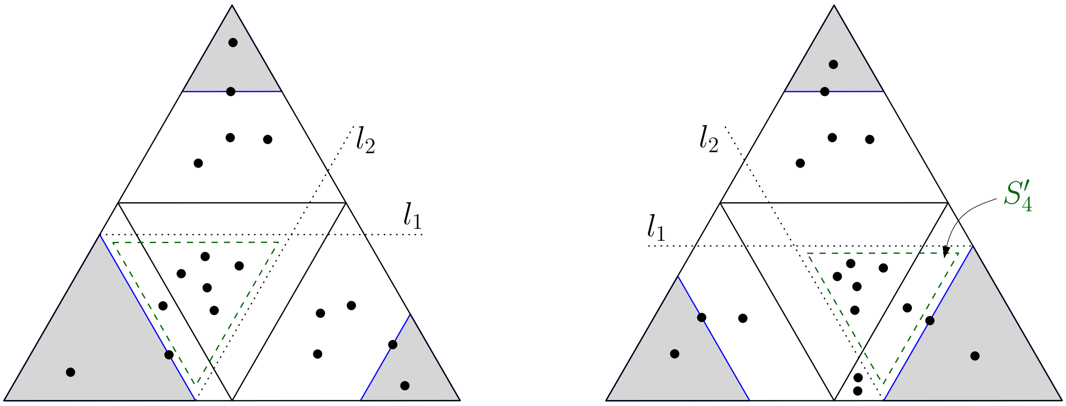
Figure 14: (a) Split S into four equal-area triangles. (b) S^{-3}_2 is larger than S^{-3}_1 and S^{-3}_4.

- *Some elements of* $\{m_1, m_2, m_4\}$ *are greater than zero.* Consider the triangles S^{-3}_1, S^{-3}_2, and S^{-3}_4. Note that the area of some of these triangles—but not all—may be equal to zero. See Figure 14(b). By induction, we get matchings of sizes at least m_1, m_2, and m_4 in S^{-3}_1, S^{-3}_2, and S^{-3}_4, respectively. Without loss of generality, assume that S^{-3}_2 is larger than S^{-3}_1 and S^{-3}_4. Consider the half-lines l_1 and l_2 that are parallel to the l^0 and l^{60} axes, and have their endpoints at the top corner and right corner of S^{-3}_2, respectively. We define S'_2 as the downward equilateral-triangle bounded by l_1, l_2, and the right side of S^{-3}_2; see the dashed triangle in Figure 14(b). Note that l_1 and l_2 do not intersect S^{-3}_1 and S^{-3}_4. In addition, S^{-3}_1, S^{-3}_2, S^{-3}_4, and S'_2 are pairwise disjoint. If any point of $S_1 \cup S_2 \cup S_3$ is to the right of l_2, then consider S^{+1}_4 and S^{-1}_{3l}. By induction, we get a matching of size at least $m_1 + m_2 + (m_3 + 1) + (m_4 + 1)$ in $S^{-3}_1 \cup S^{-3}_2 \cup S^{-1}_{3l} \cup S^{+1}_4$, and hence a matching of size at least $m+1$ in S. If any point of $S_2 \cup S_3 \cup S_4$ is above l_1, then consider S^{+1}_1 and S^{-1}_{3b}. By induction, we get a matching of size at least $(m_1 + 1) + m_2 + (m_3 + 1) + m_4$ in $S^{+1}_1 \cup S^{-3}_2 \cup S^{-1}_{3b} \cup S^{-3}_4$, and hence a matching of size at least $m + 1$ in S. Otherwise, S'_2 contains $n_3 + 3 = 4(m_3 + 1) + 2$ points. Thus, by induction, we get a matching of size at least $m_1+m_2+(m_3+2)+m_4$ in $S_1 \cup S^{-3}_2 \cup S'_2 \cup S_4$, and hence a matching of size at least $m + 1$ in S.

- $r_j = 3$, *for some* $j \in \{1, 2, 4\}$.

  Without loss of generality, assume that $r_j = r_2$. Then, $n_2 = 4m_2 + 3$. Consider the triangles S^{-3}_1, S^{-1}_2, and S^{-3}_4. See Figure 15(a). By induction, we get matchings of size at least m_1, $m_2 + 1$, and m_4 in S^{-3}_1, S^{-1}_2, and S^{-3}_4, respectively. Now we consider the largest triangle among S^{-3}_1, S^{-1}_2, and S^{-3}_4. Because of the symmetry, we have two cases: (i) S^{-1}_2 is the largest, or (ii) S^{-3}_4 is the largest.

  - *S^{-1}_2 is larger than S^{-3}_1 and S^{-3}_4.* Define the half-lines l_1, l_2, and the triangle S'_2 as in the previous case. See Figure 15(a). If any point of $S_1 \cup S_2 \cup S_3$ is to the right of l_2, then consider S^{+1}_4 and S^{-1}_{3l}. By induction, we get a matching of size at least $m_1 + (m_2 + 1) + m_3 + (m_4 + 1)$ in $S^{-3}_1 \cup S^{-1}_2 \cup S^{-1}_{3l} \cup S^{+1}_4$. If any point of $S_2 \cup S_3 \cup S_4$ is above l_1, then consider S^{+1}_1 and S^{-1}_{3b}. By induction, we get a matching of size at least $(m_1 + 1) + (m_2 + 1) + m_3 + m_4$ in $S^{+1}_1 \cup S^{-1}_2 \cup S^{-1}_{3b} \cup S^{-3}_4$. Otherwise, S'_2 contains $n_3 + 1 = 4m_3 + 2$ points. Thus, by induction, we get a matching of size at least $m_1 + (m_2 + 1) + (m_3 + 1) + m_4$ in $S_1 \cup S^{-1}_2 \cup S'_2 \cup S_4$. As a result, in all cases we get a matching of size at least $m + 1$ in S.



(a) (b)

Figure 15: (a) S_2^{-1} is larger than S_1^{-3} and S_4^{-3}. (b) S_4^{-3} is larger than S_1^{-3} and S_2^{-1}.

– *S_4^{-3} is larger than S_1^{-3} and S_2^{-1}.* Define the half-lines l_1, l_2, and the triangle S'_4 as in Figure 15(b). If any point of $S_1 \cup S_3 \cup S_4$ is above l_1, then by induction, we get a matching of size at least $(m_1+1)+(m_2+1)+m_3+m_4$ in $S_1^{+1} \cup S_2^{-1} \cup S_{3b}^{-1} \cup S_4^{+3}$. If at least three points of $S_1 \cup S_3 \cup S_4$ are to the left of l_2, then consider S_2^{+3} and S_{3r}^{-3}. Note that S_2^{+3} contains $n_2+3 = 4(m_2+1)+2$ points. By induction, we get a matching of size at least $m_1+(m_2+2)+m_3+m_4$ in $S_1^{-3} \cup S_2^{+3} \cup S_{3r}^{-3} \cup S_4^{-3}$. Otherwise, S'_4 contains at least $n_3+1 = 4m_3+2$ points. Thus, by induction, we get a matching of size at least $m_1+(m_2+1)+(m_3+1)+m_4$ in $S_1 \cup S_2 \cup S'_4 \cup S_4^{-3}$. As a result, in all cases we get a matching of size at least $m+1$ in S.

Case (ii): $R = \{0,0,1,1\}$.

In this case, we have $m = m_1+m_2+m_3+m_4$. Again, because of symmetry, we have two cases: (i) $r_3 = 0$, (ii) $r_3 \neq 0$.

- $r_3 = 0$.

  Without loss of generality assume that $r_2 = 0$ and $r_1 = r_4 = 1$. Thus, $n_1 = 4m_1+1$, $n_2 = 4m_2$, $n_3 = 4m_3$, and $n_4 = 4m_4+1$. If all elements of $\{m_1, m_2, m_4\}$ are equal to zero, then we have $m = m_3$, where $m_3 \geq 1$. Consider the triangles S_4^{+1} and S_{3l}^{-1}, that are disjoint. By induction, we get a matched pair in S_4^{+1} and a matching of size at least m_3 in S_{3l}^{-1}. Thus, in total, we get a matching of size at least $1+m_3 = m+1$ in S. Assume some elements in $\{m_1, m_2, m_4\}$ are greater than zero. Consider the triangles S_1^{-3}, S_2^{-2}, and S_4^{-3}. See Figure 16(a). By induction, we get a matching of size at least m_1, m_2, and m_4 in S_1^{-3}, S_2^{-2}, and S_4^{-3}, respectively. Now we consider the largest triangle among S_1^{-3}, S_2^{-2}, and S_4^{-3}. Because of the symmetry, we have two cases: (i) S_2^{-2} is the largest, or (ii) S_4^{-3} is the largest.

  – *S_2^{-2} is larger than S_1^{-3} and S_4^{-3}.* Define l_1, l_2, S'_2 as in Figure 16(a). If any point of $S_1 \cup S_2 \cup S_3$ is to the right of l_2, then by induction, we get a matching of size at least $m_1+m_2+m_3+(m_4+1)$ in $S_1^{-3} \cup S_2^{-2} \cup S_{3l}^{-1} \cup S_4^{+1}$. If any point of $S_2 \cup S_3 \cup S_4$ is above l_1, then by induction, we get a matching of size at least $(m_1+1)+m_2+m_3+m_4$ in $S_1^{+1} \cup S_2^{-2} \cup S_{3b}^{-1} \cup S_4^{-3}$. Otherwise, S'_2 contains $n_3+2 = 4m_3+2$ points. Thus, by induction, we get a matching of size at least $m_1+m_2+(m_3+1)+m_4$ in $S_1 \cup S_2^{-2} \cup S'_2 \cup S_4$. In all cases we get a matching of size at least $m+1$ in S.

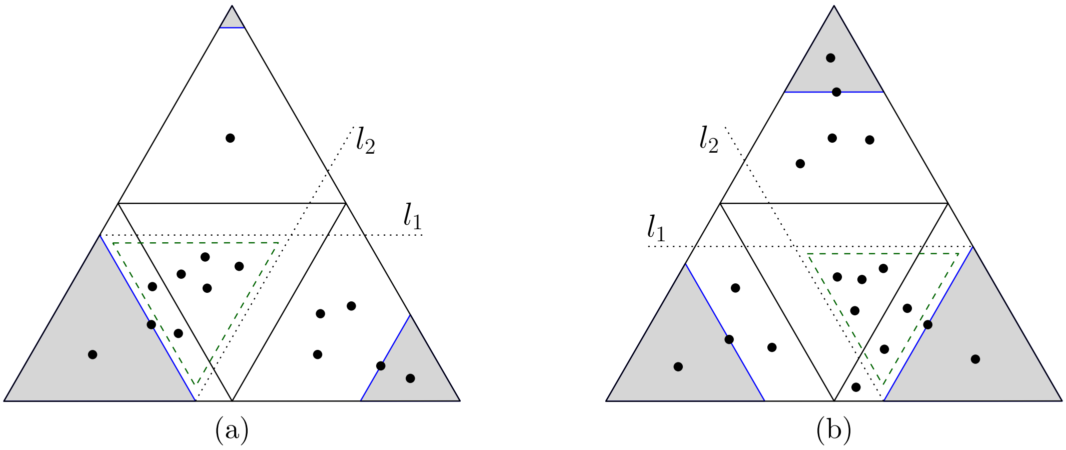


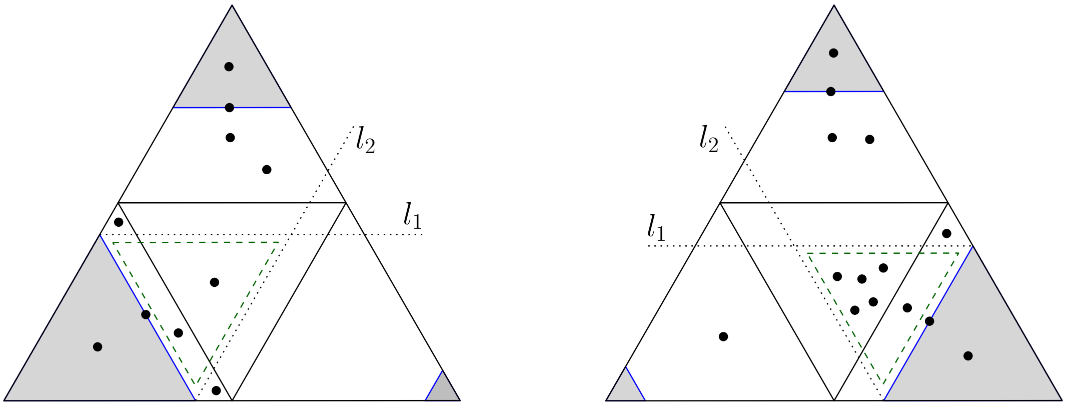
Figure 16: (a) S_2^{-2} is larger than S_1^{-3} and S_4^{-3}. (b) S_4^{-3} is larger than S_1^{-3} and S_2^{-2}.

- *S_4^{-3} is larger than S_1^{-3} and S_2^{-2}.* Define l_1, l_2, S'_4 as in Figure 16(b). If any point of $S_1 \cup S_3 \cup S_4$ is above l_1, then by induction, we get a matching of size at least $(m_1+1)+m_2+m_3+m_4$ in $S_1^{+1} \cup S_2^{-2} \cup S_{3b}^{-1} \cup S_4^{+3}$. If at least two points of $S_1 \cup S_3 \cup S_4$ are to the left of l_2, then by induction, we get a matching of size at least $m_1+(m_2+1)+m_3+m_4$ in $S_1^{-3} \cup S_2^{+2} \cup S_{3r}^{-2} \cup S_4^{-3}$. Otherwise, S'_4 contains at least $n_3+2 = 4m_3+2$ points. Thus, by induction, we get a matching of size at least $m_1+m_2+(m_3+1)+m_4$ in $S_1 \cup S_2 \cup S'_4 \cup S_4^{-3}$. In all cases we get a matching of size at least $m+1$ in S.

- $r_3 \neq 0$.

  In this case $r_3 = 1$, and without loss of generality, assume that $r_2 = 1$; that means $r_1 = r_4 = 0$. Thus, $n_1 = 4m_1$, $n_2 = 4m_2+1$, $n_3 = 4m_3+1$, and $n_4 = 4m_4$. If all elements of $\{m_1, m_2, m_4\}$ are equal to zero, then we have $m = m_3$, where $m_3 \geq 1$. Consider the triangles S_2^{+1} and S_{3r}^{-1}, that are disjoint. By induction, we get a matched pair in S_2^{+1} and a matching of size at least m_3 in S_{3r}^{-1}. Thus, in total, we get a matching of size at least $1+m_3 = m+1$ in S. Assume some elements in $\{m_1, m_2, m_4\}$ are greater than zero. Consider the triangles S_1^{-2}, S_2^{-3}, and S_4^{-2}. See Figure 17(a). By induction, we get matchings of size at least m_1, m_2, and m_4 in S_1^{-2}, S_2^{-3}, and S_4^{-2}, respectively. Now we consider the largest triangle among S_1^{-2}, S_2^{-3}, and S_4^{-2}. Because of symmetry, we have two cases: (i) S_2^{-3} is the largest, or (ii) S_4^{-2} is the largest.

  - *S_2^{-3} is larger than S_1^{-2} and S_4^{-2}.* Define l_1, l_2, S'_2 as in Figure 17(a). If at least two points of $S_1 \cup S_2 \cup S_3$ are to the right of l_2, then by induction, we get a matching of size at least $m_1+m_2+m_3+(m_4+1)$ in $S_1^{-2} \cup S_2^{-3} \cup S_{3l}^{-2} \cup S_4^{+2}$. If at least two points of $S_2 \cup S_3 \cup S_4$ are above l_1, then by induction, we get a matching of size at least $(m_1+1)+m_2+m_3+m_4$ in $S_1^{+2} \cup S_2^{-3} \cup S_{3b}^{-2} \cup S_4^{-2}$. Otherwise, S'_2 contains $n_3+1 = 4m_3+2$ points, and we get a matching of size at least $m_1+m_2+(m_3+1)+m_4$ in $S_1 \cup S_2^{-3} \cup S'_2 \cup S_4$. In all cases we get a matching of size at least $m+1$ in S.
  - *S_4^{-2} is larger than S_1^{-2} and S_2^{-3}.* Define l_1, l_2, S'_4 as in Figure 17(b). If at least two points of $S_2 \cup S_3 \cup S_4$ are above l_1, then by induction, we get a matching of size at least $(m_1+1)+m_2+m_3+m_4$ in $S_1^{+2} \cup S_2^{-3} \cup S_{3b}^{-2} \cup S_4^{-2}$. If any point of $S_1 \cup S_3 \cup S_4$ is to the left of l_2, then by induction, we get a matching of size at least $m_1+(m_2+1)+m_3+m_4$ in $S_1^{-2} \cup S_2^{+1} \cup S_{3r}^{-1} \cup S_4^{-2}$. Otherwise, S'_4 contains at least $n_3+1 = 4m_3+2$ points, and



(a) (b)

Figure 17: (a) S_2^{-3} is larger than S_1^{-2} and S_4^{-2}. (b) S_4^{-2} is larger than S_1^{-2} and S_2^{-3}.

we get a matching of size at least $m_1 + m_2 + (m_3 + 1) + m_4$ in $S_1 \cup S_2 \cup S'_4 \cup S_4^{-2}$. In all cases we get a matching of size at least $m + 1$ in S.

□

8 A Conjecture on Strong Matchings in $G_\ominus(P)$

In this section, we discuss a possible way to further improve upon Theorem 2, which says that Algorithm 1 computes a strong matching of size at least $\lceil \frac{n-1}{17} \rceil$ in $G_\ominus(P)$. We also discuss a construction leading to the conjecture that Algorithm 1 computes a strong matching of size at least $\lceil \frac{n-1}{8} \rceil$ in $G_\ominus(P)$; unfortunately we are not able to prove this.

In Section 4 we proved that $\mathcal{I}(e^+)$ contains at most 16 edges. In order to achieve this upper bound we used the fact that the centers of the disks in $\mathcal{I}(e^+)$ are far apart. We did not consider the endpoints of the edges representing these disks. By Observation 4, the disks representing the edges in $\mathcal{I}(e^+)$ cannot contain any of the endpoints. We applied this observation only on u and v. Unfortunately, our attempts to apply this observation on the endpoints of edges in $\mathcal{I}(e^+)$ have been so far unsuccessful.

Recall that T is a Euclidean minimum spanning tree of P, and for every edge $e = (u, v)$ in T, $\deg(e)$ is the degree of e in $T(e^+)$, where $T(e^+)$ is the set of all edges of T with weight at least $w(e)$. Note that $w(e)$ is directly related to the Euclidean distance between u and v. Observe that the discs representing the edges adjacent to e intersect $D(u, v)$. Thus, these edges are in $\text{Inf}(e)$. We call an edge e in T a *minimal edge* if e is not longer than any of its adjacent edges. We observed that:

Conjecture 1. $\text{Inf}(T)$ *is at most the maximum degree of a minimal edge.*

Monma and Suri [12] showed that for every point set P there exists a Euclidean minimum spanning tree, $MST(P)$, of maximum vertex degree five. Thus, the maximum edge degree in $MST(P)$ is 9. We show that for every point set P, there exists a Euclidean minimum spanning tree, $MST(P)$, such that the degree of each node is at most five and the degree of each minimal edge is at most eight. This would imply the conjecture that $\text{Inf}(MST(P)) \leq 8$. That is, Algorithm 1 would return a strong matching of size at least $\lceil \frac{n-1}{8} \rceil$.

Lemma 15. *If uv and uw are two adjacent edges in $MST(P)$, then the triangle $\triangle uvw$ has no point of $P \setminus \{u, v, w\}$ in its interior or on its boundary.*

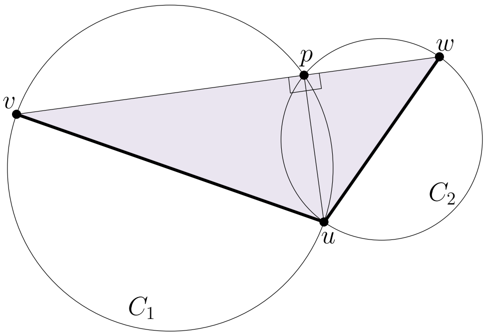


Figure 18: In $MST(P)$, the triangle $\triangle uvw$ formed by two adjacent edges uv and uw, is empty.

Proof. If the angle between uv and uw is equal to π, then there is no other point of P on uv and uw. Assume that $\angle vuw < \pi$. Refer to Figure 18. Since $MST(P)$ is a subgraph of the Gabriel graph, the circles C_1 and C_2 with diameters uv and uw are empty. Since $\angle vuw < \pi$, C_1 and C_2 intersect each other at two points, say u and p. Connect u, v and w to p. Since uv and uw are the diameters of C_1 and C_2, $\angle upv = \angle wpu = \pi/2$. This means that vw is a straight line segment. Since C_1 and C_2 are empty and $\triangle uvw \subset C_1 \cup C_2$, it follows that $\triangle uvw \cap P = \{u, v, w\}$. □

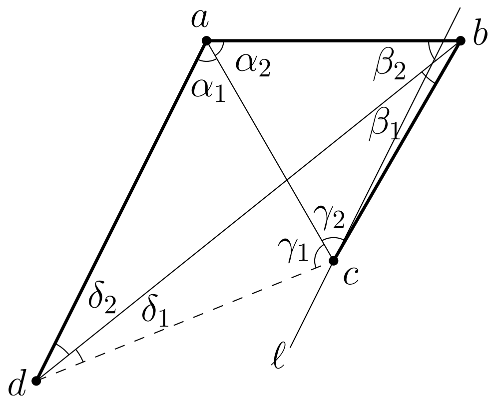


Figure 19: Illustration of Lemma 16: $|ab| \leq |bc| \leq |ad|$, $\angle abc \geq \pi/3$, $\angle bad \geq \pi/3$, and $\angle abc + \angle bad \leq \pi$.

Lemma 16. *Follow Figure 19. For a convex-quadrilateral $Q = a, b, c, d$ with $|ab| \leq |bc| \leq |ad|$, if $\min\{\angle abc, \angle bad\} \geq \pi/3$ and $\angle abc + \angle bad \leq \pi$, then $|cd| \leq |ad|$.*

Proof. Let $\alpha_1 = \angle cad$, $\alpha_2 = \angle bac$, $\beta_1 = \angle cbd$, $\beta_2 = \angle abd$, $\gamma_1 = \angle acd$, $\gamma_2 = \angle acb$, $\delta_1 = \angle bdc$, and $\delta_2 = \angle adb$; see Figure 19. Since $|ab| \leq |bc| \leq |ad|$,

$$\gamma_2 \leq \alpha_2 \text{ and } \delta_2 \leq \beta_2.$$

Let ℓ be a line passing through c that is parallel to ad. Since $\angle abc + \angle bad \leq \pi$, ℓ intersects the line segment ab. This implies that $\alpha_1 \leq \gamma_2$. If $\beta_1 < \delta_1$, then $|cd| < |bc|$, and hence $|cd| < |ad|$ and we are done. Assume that $\delta_1 \leq \beta_1$. In this case, $\delta \leq \beta$. Now consider the two triangles $\triangle abc$ and $\triangle acd$. Since $\delta \leq \beta$ and $\alpha_1 \leq \gamma_2$, $\alpha_2 \leq \gamma_1$. Then we have

$$\alpha_1 \leq \gamma_2 \leq \alpha_2 \leq \gamma_1.$$

Since $\alpha_1 \leq \gamma_1$, $|cd| \leq |ad|$, where the equality holds only if $\alpha_1 = \gamma_2 = \alpha_2 = \gamma_1$, i.e., Q is a diamond. This completes the proof. $\square$

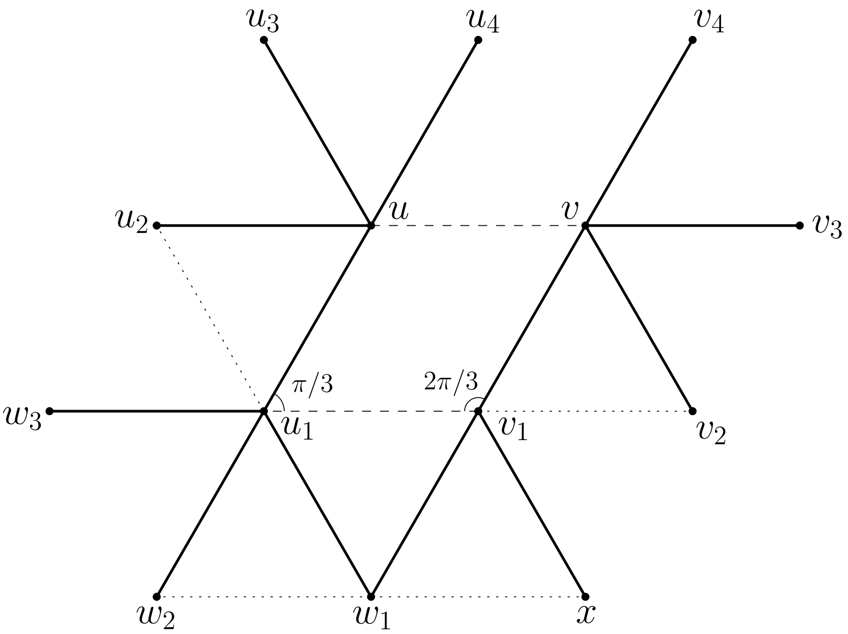


Figure 20: Solid segments represent the edges of $MST(P)$. Dashed segments represent the swapped edges. Dotted segments represent the edges that cannot exist.

Lemma 17. *Every finite set of points P in the plane admits a minimum spanning tree whose node degree is at most five and whose minimal-edge degree is at most eight.*

Proof. Consider a minimum spanning tree, $MST(P)$, of maximum vertex degree 5. The maximum edge degree in $MST(P)$ is 9. Consider any minimal edge, uv. If the degree of uv is 8, then $MST(P)$ satisfies the statement of the lemma. Assume that the degree of uv is 9. Let u_1, u_2, u_3, u_4 and v_1, v_2, v_3, v_4 be the the neighbors of u and v in clockwise and counterclockwise orders, respectively. See Figure 20. In $MST(P)$, the angles between two adjacent edges are at least $\pi/3$. Since $\angle u_i u u_{i+1} \geq \pi/3$ and $\angle v_i v v_{i+1} \geq \pi/3$ for $i = 1, 2, 3$, either $\angle vuu_1 + \angle uvv_1 \leq \pi$ or $\angle vuu_4 + \angle uvv_4 \leq \pi$. Without loss of generality assume that $\angle vuu_1 + \angle uvv_1 \leq \angle vuu_4 + \angle uvv_4$ and $\angle vuu_1 + \angle uvv_1 \leq \pi$. We prove that the spanning tree obtained by swapping the edge uv with u_1v_1 is also a minimum spanning tree, and it has one fewer minimal-edge of degree 9. By repeating this procedure at each minimal-edge of degree 9, we obtain a minimum spanning tree that satisfies the statement of the lemma. Let $Q = u, v, v_1, u_1$. By Lemma 15, v_1 is outside the triangle $\triangle u_1 uv$, and u_1 is outside the triangle $\triangle uvv_1$. In addition, u_1 and v_1 are on the same side of the line subtended from uv. Thus, Q is a convex quadrilateral. Without loss of generality assume that $|vv_1| \leq |uu_1|$. By Lemma 16, $|u_1v_1| \leq |uu_1|$. If $|u_1v_1| < |uu_1|$, we get a contradiction to Lemma 2. Thus, assume that $|u_1v_1| = |uu_1|$. As shown in the proof of Lemma 16, this case happens only when Q is a diamond. This implies that $\angle vuu_1 + \angle uvv_1 = \pi$, and consequently $\angle vuu_4 + \angle uvv_4 = \pi$. In addition, $\angle u_i u u_{i+1} = \pi/3$ and $\angle v_i v v_{i+1} = \pi/3$ for $i = 1, 2, 3$. To establish the validity of our edge-swap, observe that the nine edges incident to u and v are all equal in length. Therefore, swapping uv with u_1v_1 does not change the cost of the spanning tree and, furthermore, the resulting tree is a valid spanning tree since u_1v_1 is not an edge of the original spanning tree $MST(P)$; otherwise u, v, v_1, and u_1 would form a cycle. We have removed a minimal edge uv of degree 9, but it remains to show that the degree of u_1 and

v_1 does not increase to six and new minimal edge of degree 9 is not generated. Note that u_1u_2 and v_1v_2 are not the edges of $MST(P)$, and hence, $\deg(u_1)$ and $\deg(v_1)$ are still less than six. In order to show that no new minimal edge is generated, we differentiate between two cases:

- $\min\{\angle vv_1u_1, \angle v_1u_1u\} > \pi/3$. Since $\angle v_1u_1u > \pi/3$ and $\angle uu_1u_2 = \pi/3$, u_1 can be adjacent to at most two vertices other than u and v_1, and hence $\deg(u_1) \leq 4$; similarly $\deg(v_1) \leq 4$. Thus, u, v, u_1, and v_1 are of degree at most four, and hence no new minimal edge of degree 9 is generated.

- $\min\{\angle vv_1u_1, \angle v_1u_1u\} = \pi/3$. Without loss of generality assume that $\angle v_1u_1u = \pi/3$. This implies that $\angle vv_1u_1 = 2\pi/3$. Since $\angle v_1u_1u = \pi/3$ and $\angle uu_1u_2 = \pi/3$, u_1 is adjacent to at most three vertices other than u and v_1. Let u, v_1, w_1, w_2, w_3 be the neighbors of u_1 in clockwise order. Note that v_1 is not adjacent to u, v_2 nor w_1. But v_1 can be connected to another vertex, say x, which implies that $\deg(v_1) \leq 3$. We prove that the spanning tree obtained by swapping the edge u_1v_1 with v_1w_1 is also a minimum spanning tree of node degree at most five, that has one fewer minimal edge of degree 9. The new tree is a legal minimum spanning tree for P, because $|v_1w_1| = |v_1u_1|$. In addition, $\deg(u_1) \leq 4$ and $\deg(v_1) \leq 4$. Since w_1w_2 and w_1x are illegal edges, $\deg(w_1) \leq 4$. Thus, u, v, u_1, v_1, and w_1 are of degree at most four and no new minimal edge of degree 9 is generated. This completes the proof that our edge swap reduces the number of minimal edges of degree nine by one.

□

9 Conclusion

Given a set of n points in general position in the plane, we considered the problem of strong matching of points with convex geometric shapes. A matching is strong if the objects representing whose edges are pairwise disjoint. In this paper we presented algorithms that compute strong matchings of points with diametral disks, equilateral triangles, and squares. Specifically we showed that:

- There exists a strong matching of points with diametral-disks of size at least $\lceil \frac{n-1}{17} \rceil$.
- There exists a strong matching of points with downward equilateral-triangles of size at least $\lceil \frac{n-1}{9} \rceil$.
- There exists a strong matching of points with downward/upward equilateral-triangles of size at least $\lceil \frac{n-1}{4} \rceil$.
- There exists a strong matching of points with axis-parallel squares of size at least $\lceil \frac{n-1}{4} \rceil$.

The existence of a downward/upward equilateral-triangle matching of size at least $\lceil \frac{n-1}{4} \rceil$, implies the existence of either a downward equilateral-triangle matching of size at least $\lceil \frac{n-1}{8} \rceil$ or an upward equilateral-triangle matching of size at least $\lceil \frac{n-1}{8} \rceil$. This does not, however, imply a lower bound better than $\lceil \frac{n-1}{9} \rceil$ for downward equilateral-triangle matching (or any fixed oriented equilateral-triangle).

A natural open problem is to improve any of the provided lower bounds, or extend these results for other convex shapes. A specific open problem is to prove that Algorithm 1 computes a strong matching of points with diametral-disks of size at least $\lceil \frac{n-1}{8} \rceil$ as discussed in Section 8.

Acknowledgments

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