An Improved Construction for Spanners of Disks

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June 26, 2020

Abstract

Let $\mathcal{D}$ be a set of n pairwise disjoint disks in the plane. Consider the metric space in which the distance between any two disks D and D' in $\mathcal{D}$ is the length of the shortest line segment that connects D and D'. For any real number $\varepsilon > 0$, we show how to obtain a $(1+\varepsilon)$-spanner for this metric space that has at most $(2\pi/\varepsilon) \cdot n$ edges. The previously best known result is by Bose *et al.* (Journal of Discrete Algorithms, 2011). Their $(1+\varepsilon)$-spanner is a variant of the Yao graph and has at most $(8\pi/\varepsilon) \cdot n$ edges. Our new spanner is also a variant of the Yao graph.

1 Introduction

Let $\mathcal{D}$ be a set of n closed disks in the plane that are pairwise disjoint. Assume that inside any disk, we can travel at infinite speed, whereas outside all disks, we can travel at unit-speed. For two disks D and D' in $\mathcal{D}$, a point x in D, and a point y in D', let $\delta(x, y)$ be the minimum amount of time needed to travel from x to y. For any point x' in D and any point y' in D', we have $\delta(x', y') = \delta(x, y)$. Thus, we can write this common value as $\delta(D, D')$: It is the minimum amount of time needed to travel from any point in D to any point in D'.

We can regard every disk to be a "safe" region and the area outside all disks to be an "unsafe" region. Then, $\delta(D, D')$ is the minimum amount of time spent in the unsafe region when traveling from disk D to disk D'.

For any disk D in $\mathcal{D}$, denote its center and radius by $center(D)$ and $radius(D)$, respectively. We assume that $radius(D) \geq 0$; in case $radius(D) = 0$, the disk D is a single point. For any two points x and y in the plane, denote their Euclidean distance by $|xy|$. The distance $dist(D, D')$ between two disks D and D' in $\mathcal{D}$ is defined to be

$$dist(D, D') = \min\{|xy| : x \in D \text{ and } y \in D'\}. \quad (1)$$

Thus, $dist(D, D) = 0$, whereas, for $D \neq D'$, $dist(D, D')$ is the length of the shortest line segment joining D and D', i.e.,

$$dist(D, D') = |center(D), center(D')| - radius(D) - radius(D').$$

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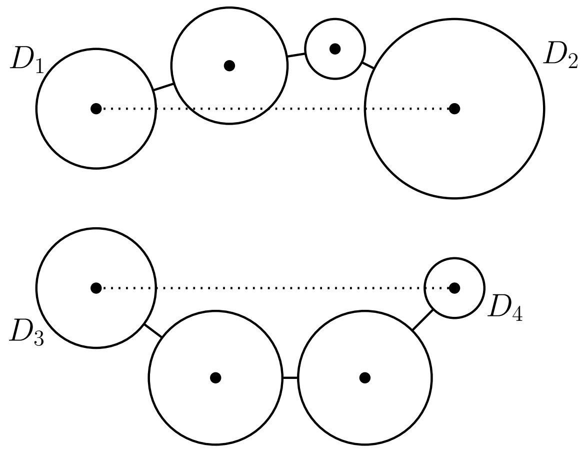

Figure 1: The disks D_1 and D_2 are not visible, $\delta_T(D_1, D_2)$ is equal to the sum of the three solid line segments in the top of the figure. The disks D_3 and D_4 are visible, $\delta_T(D_3, D_4)$ is equal to the sum of the three solid line segments in the bottom of the figure.

We say that two distinct disks D and D' in $\mathcal{D}$ are *visible*, if the line segment joining their centers does not intersect the interior of any other disk in $\mathcal{D}$.

Consider an optimal path between two distinct disks of $\mathcal{D}$, and a portion of this path that is outside all disks. This portion starts at a disk, say D, and ends at another disk, say D'. It is clear that these two disks are visible and that the length of this portion is equal to $dist(D, D')$.

Based on this, we define the following *transportation graph* T:

- The vertex set of T is the set $\mathcal{D}$.
- For any two distinct disks D and D' in $\mathcal{D}$, T contains the undirected edge $\{D, D'\}$ if and only if D and D' are visible. The length (or weight) of this edge is equal to $dist(D, D')$.

For any two disks D and D' in $\mathcal{D}$, let $\delta_T(D, D')$ denote their shortest-path distance in T; see Figure 1. Observe that this distance is equal to the value $\delta(D, D')$ that was introduced at the beginning of this section.

The edge lengths in T may not satisfy the triangle inequality: If $\{D, D'\}$ is an edge in T, then this edge is not necessarily a shortest path in T between D and D'; the bottom part of Figure 1 gives an example. However, the shortest-path distances $\delta_T(D, D')$ do satisfy the triangle inequality.

Intuitively, for any two disks D and D', $\delta_T(D, D') \leq dist(D, D')$, because any path in T between D and D' of length more than $dist(D, D')$ can be shortened by making shortcuts. (We will formally prove this in Lemma 1.) As Figure 1 shows, $\delta_T(D, D')$ can be much smaller than $dist(D, D')$, even if the disks D and D' are visible.

A pair $\{D, D'\}$ is called a *useful edge* in T, if $\{D, D'\}$ is an edge in T and $\delta_T(D, D') = dist(D, D')$. If $\{D, D'\}$ is an edge in T and $\delta_T(D, D') < dist(D, D')$, then $\{D, D'\}$ is called a *useless edge* in T.

Even though the transportation graph T contains all information needed to determine shortest-paths, it may be a complete graph: Consider n points in the plane such that no three of them are on a line. Then, for a sufficiently small $\varepsilon > 0$, the disks of radii ε that are centered at these points lead to a complete transportation graph.

In this paper, we consider the problem of obtaining a sparse subgraph G of T, such that shortest-path distances in G are within a small multiplicative factor of the corresponding shortest-path distances in T. We denote the shortest-path distance in G between two disks D and D' by $\delta_G(D, D')$.

Definition 1 Let $\mathcal{D}$ be a set of n closed disks in the plane that are pairwise disjoint, and let T be the corresponding transportation graph. Let G be a subgraph of T, and let $t \geq 1$ be a real number. The graph G is a *t-spanner* of T, if for any two disks D and D' in $\mathcal{D}$,

$$\delta_G(D, D') \leq t \cdot \delta_T(D, D'). \tag{2}$$

The *stretch factor* of G is the smallest t for which G is a t-spanner of T.

1.1 Previous Work

Our notion of a spanner is almost equivalent to spanners of additively weighted point sets, as introduced by Bose *et al.* [1]: Let S be a set of n points in the plane, and let each point p of S have a weight $\omega(p) \geq 0$. Define the additively weighted distance between two distinct points p and q to be $|pq| - \omega(p) - \omega(q)$. Replace each point p of S by a disk with center p and radius $\omega(p)$. If the resulting disks are pairwise disjoint, then the additively weighted distance between two points of S is the same as the distance defined in (1) for the corresponding disks.

Bose *et al.* [1] obtained the following result: Consider a sufficiently small real number $\theta > 0$, such that $2\pi/\theta$ is an integer, and assume that the disks described above are pairwise disjoint. There exists a t-spanner of the additively weighted point set S, where

$$t = \frac{1}{\cos(2\theta) - \sin(2\theta)}.$$

This spanner has at most $(4\pi/\theta) \cdot n$ edges. If we replace Lemma 1 in Bose *et al.* [1] by Lemma 6 below, then we obtain the slightly better upper bound of

$$t = \frac{1}{1 - 2\sin\theta} = 1 + 2\theta + O(\theta^2) \tag{3}$$

on the stretch factor.

We describe the approach of Bose *et al.* Instead of the weighted points of the set S, we will consider the corresponding set of disks. We assume that these disks are pairwise disjoint. For each disk D, cover the plane by $2\pi/\theta$ cones of angle θ, all having their apex at $center(D)$. We will refer to the bounding rays of a cone as the *legs* of this cone.

The first idea is to consider each such cone C and connect D by an edge to the disk D' whose center is in C and for which $dist(D, D')$ is minimum. Bose *et al.* show that this graph can have an arbitrarily large stretch factor. Instead, they do the following, for each disk D and each cone C with apex at $center(D)$:

1. If there is a disk whose center is in C and that intersects both legs of C, then let D' be such a disk for which $dist(D, D')$ is minimum. The edge $\{D, D'\}$ is added to the spanner.

2. If there is a disk D' whose center is in C and for which $dist(D, D') \geq radius(D)$, then let D' be such a disk for which $dist(D, D')$ is minimum. The edge $\{D, D'\}$ is added to the spanner.

The proof of Bose *et al.* and Lemma 6 below imply that the stretch factor of the resulting graph is bounded from above by the value in (3). Observe that, since at most two edges are added to the spanner for each disk and each cone, the total number of edges is at most $(4\pi/\theta) \cdot n$.

We remark that this spanner may contain edges between pairs of disks that are not visible. However, if we consider, in 1. and 2. above, only disks D' such that D and D' are visible, then we do obtain a subgraph of the transportation graph T and the stretch factor of the graph is still bounded from above by the value in (3).

If each disk has a radius of zero, then the spanner problem becomes the standard spanner problem for point sets. In this case, the spanner of Bose *et al.* is equal to the well known Yao graph; see Flinchbaugh and Jones [4] and Yao [6]. The stretch factor of the Yao graph is at most $1/(1 - 2\sin(\theta/2))$; see the appendix in Bose *et al.* [2].

There are many other results known for constructing spanners of point sets. We refer the reader to the survey by Eppstein [3] and the book by Narasimhan and Smid [5].

To the best of our knowledge, [1] is the only paper that contains results for disk spanners.

1.2 Our Result

Recall that the spanner of Bose *et al.* [1] adds at most two edges for each disk and each cone having its apex at the center of the disk. In this paper, we show that it is enough to add only one edge. Additionally, we show that the stretch factor of our spanner is smaller than the value in (3). We will prove the following result:

Theorem 1 *Let $\mathcal{D}$ be a set of n closed disks in the plane that are pairwise disjoint, consider the transportation graph T of $\mathcal{D}$, and let $\theta > 0$ be a sufficiently small real number such that $2\pi/\theta$ is an integer. There exists a t-spanner of T, where*

$$t = \frac{1 - 2\sin\theta}{1 - 2\sin\theta - 2(1 - \sin\theta)\sin(\theta/2)}.$$

The number of edges in this spanner is at most $(2\pi/\theta) \cdot n$. For θ approaching zero, $t = 1 + \theta + O(\theta^2)$.

This result is valid for any value of θ for which the denominator of t is positive. This is the case when $\theta < 0.389$ radians.

Let $\varepsilon > 0$ be a small real number. To obtain a $(1+\varepsilon)$-spanner of T, we can take θ to be slightly less than ε. The resulting spanner contains at most $(2\pi/\varepsilon) \cdot n$ edges. In contrast, to obtain the same stretch factor using the spanner by Bose *et al.*, we have to take θ to be slightly smaller than $\varepsilon/2$ and the number of edges is at most $(8\pi/\varepsilon) \cdot n$.

The rest of this paper is organized as follows. In Section 2, we prove two general results for the transportation graph T. Our new spanner is defined in Section 3, whereas the analysis of its stretch factor is given in Section 4. Section 5 concludes the paper with some future research directions.

2 Preliminaries

Let $\mathcal{D}$ be a set of n closed disks in the plane that are pairwise disjoint, and consider the transportation graph T of $\mathcal{D}$. We start by formally proving that the shortest-path distance between any pair of disks D and D' is at most $dist(D, D')$.

Lemma 1 *For any two disks D and D' in $\mathcal{D}$, $\delta_T(D, D') \leq dist(D, D')$.*

Proof. Consider the sequence of $\binom{n}{2}$ pairs $\{D, D'\}$ of distinct disks in $\mathcal{D}$, sorted in non-decreasing order of their lengths $dist(D, D')$. The proof is by induction on the rank of $dist(D, D')$ in this sequence.

If $dist(D, D')$ is minimum, then D and D' are visible and, thus, $\{D, D'\}$ is an edge in T. In this case, $\delta_T(D, D') = dist(D, D')$.

For the induction step, assume that $dist(D, D')$ is not minimum. If D and D' are visible, then $\{D, D'\}$ is an edge in T and, therefore, $\delta_T(D, D') \leq dist(D, D')$. Assume that D and D' are not visible. Let L be the line segment between $center(D)$ and $center(D')$, and let $D = D_1, D_2, \ldots, D_k = D'$ be the disks whose interiors intersect L; the indices refer to the order in which we encounter these disks when walking along L from $center(D)$ to $center(D')$. For each i with $1 \leq i < k$, let ℓ_i denote the length along L from D_i to D_{i+1}. Observe that D_i and D_{i+1} may not be visible. However, since

$$dist(D_i, D_{i+1}) \leq \ell_i < dist(D, D'),$$

we have, by induction,

$$\delta_T(D_i, D_{i+1}) \leq dist(D_i, D_{i+1}).$$

It follows that

$$\delta_T(D, D') \leq \sum_{i=1}^{k-1} \delta_T(D_i, D_{i+1}) \leq \sum_{i=1}^{k-1} dist(D_i, D_{i+1}) \leq \sum_{i=1}^{k-1} \ell_i < dist(D, D').$$

■

Recall that an edge $\{D, D'\}$ in the transportation graph T is called useful in T, if $\delta_T(D, D') = dist(D, D')$. The definition of t-spanner requires that (2) holds for any two disks in T. The following lemma states that it is sufficient for (2) to hold for every useful edge in T.

Lemma 2 *Let G be a subgraph of T and let $t \geq 1$ be a real number. The following two statements are equivalent:*

1. *G is a t-spanner of T.*
2. *For every useful edge $\{D, D'\}$ in T, $\delta_G(D, D') \leq t \cdot \delta_T(D, D')$.*

Proof. It is clear that the first statement implies the second one. To prove the converse, assume that the second statement holds. Let D and D' be two disks in $\mathcal{D}$. If $D = D'$, then $\delta_G(D, D') = \delta_T(D, D') = 0$ and (2) holds. Assume that $D \neq D'$. Let $D = D_1, D_2, \ldots, D_k = D'$ be a shortest path in T between D and D'. For each i with $1 \leq i < k$, $\{D_i, D_{i+1}\}$ is a useful edge in T and, thus

$$\delta_G(D_i, D_{i+1}) \leq t \cdot \delta_T(D_i, D_{i+1}).$$

It follows that

$$\delta_G(D, D') \leq \sum_{i=1}^{k-1} \delta_G(D_i, D_{i+1}) \leq t \sum_{i=1}^{k-1} \delta_T(D_i, D_{i+1}) = t \cdot \delta_T(D, D'),$$

i.e., (2) holds. ∎

3 The Spanner G_θ

Let $\mathcal{D}$ be a set of n closed disks in the plane that are pairwise disjoint, and let $\theta > 0$ be a sufficiently small real number such that $2\pi/\theta$ is an integer. The graph G_θ is defined as follows; refer to Figure 2.

1. The vertex set of G_θ is equal to $\mathcal{D}$.
2. The edges of G_θ are obtained in the following way: For every disk D in $\mathcal{D}$, cover the plane by $2\pi/\theta$ cones of angle θ, all having their apex at $center(D)$. For each such cone C, let $\mathcal{D}(D, C)$ be the set of all disks D' in $\mathcal{D}$ such that
   - $D' \neq D$,
   - $center(D')$ is in the cone C,
   - D and D' are visible,
   - and $radius(D') \geq radius(D)$.

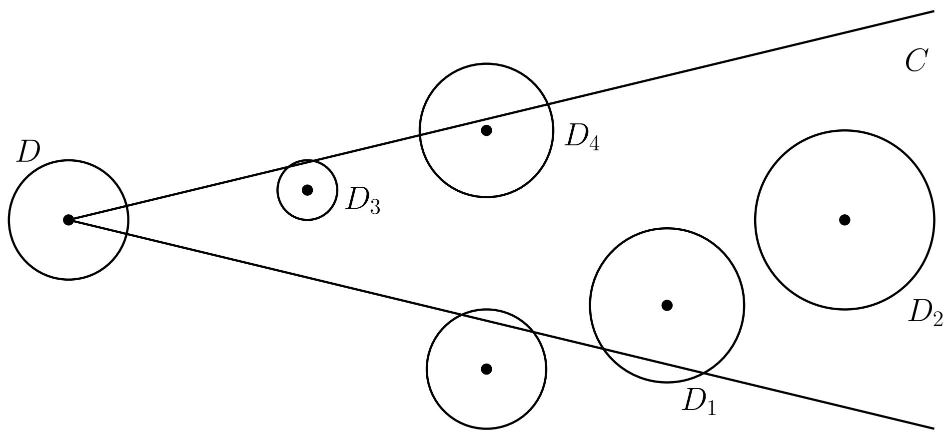

Figure 2: The set $\mathcal{D}(D, C)$ consists of the two disks D_1 and D_2. Since $dist(D, D_1) < dist(D, D_2)$, the graph G_θ contains the edge $\{D, D_1\}$.

If $\mathcal{D}(D, C) = \emptyset$, then there is no edge in G_θ corresponding to D and C. Otherwise, let D' be a disk in $\mathcal{D}(D, C)$ for which $dist(D, D')$ is minimum. (Ties are broken arbitrarily.) The graph G_θ contains the undirected edge $\{D, D'\}$.

Observe that the graph G_θ may contain edges $\{D, D'\}$ with $\delta_{G_\theta}(D, D') < dist(D, D')$. Such edges are useless, in the sense that they are not on any shortest path in G_θ. The following lemma is obvious.

Lemma 3 *The graph G_θ contains at most $(2\pi/\theta) \cdot n$ edges.*

In the next section, we will prove an upper bound on the stretch factor of G_θ. This, together with Lemma 3, will prove Theorem 1.

4 The Stretch Factor of the Graph G_θ

Throughout this section, $\theta > 0$ denotes a sufficiently small real number such that $2\pi/\theta$ is an integer. Let

$$t = \frac{1 - 2\sin\theta}{1 - 2\sin\theta - 2(1 - \sin\theta)\sin(\theta/2)}.$$

We will prove that the graph G_θ is a t-spanner of T. By Lemma 2, it is sufficient to show that

$$\delta_{G_\theta}(D, D') \leq t \cdot \delta_T(D, D') \tag{4}$$

for every useful edge $\{D, D'\}$ in T.

The proof is by induction on the rank of the length $dist(D, D')$ in the sequence of all useful edges in T.

The base case is when $\{D, D'\}$ is a useful edge in T having minimum length $dist(D, D')$. Observe that the disks D and D' are visible. It follows from the definition of the graph G_θ

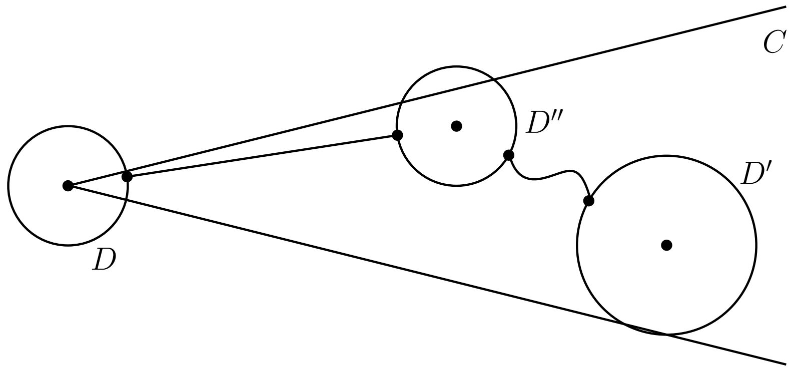

Figure 3: The path in G_θ between D and D' consists of the edge $\{D, D''\}$, followed by a shortest path in G_θ between D'' and D'.

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that $\{D, D'\}$ is an edge in G_θ and $\delta_T(D, D') = dist(D, D') = \delta_{G_\theta}(D, D)$. Therefore, (4) holds.

For the induction step, let $\{D, D'\}$ be a useful edge in T that does not have minimum weight. We assume that $\delta_{G_\theta}(D_1, D_2) \leq t \cdot \delta_T(D_1, D_2)$ for every useful edge $\{D_1, D_2\}$ in T with $dist(D_1, D_2) < dist(D, D')$. Our goal is to prove that (4) holds for $\{D, D'\}$.

If $\{D, D'\}$ is an edge in G_θ, then, since $\{D, D'\}$ is a useful edge in T,

$$dist(D, D') = \delta_T(D, D') \leq \delta_{G_\theta}(D, D') \leq dist(D, D').$$

Thus, $\delta_{G_\theta}(D, D') = \delta_T(D, D')$ and (4) holds.

In the rest of this section, we assume that $\{D, D'\}$ is not an edge in G_θ. Recall that D and D' are visible. We assume, without loss of generality, that

$$radius(D') \geq radius(D).$$

Let C be the cone with apex $center(D)$ that contains $center(D')$. By the definition of the graph G_θ, there exists an edge $\{D, D''\}$ in G_θ such that $center(D'')$ is in the cone C, the disks D and D'' are visible,

$$radius(D'') \geq radius(D)$$

and

$$dist(D, D'') \leq dist(D, D').$$

Observe that $\{D, D''\}$ may be a useless edge in T.

Our proof that (4) holds for the edge $\{D, D'\}$ consists of the following steps. Refer to Figure 3.

1. We obtain a path in G_θ between D and D', by first traversing the edge $\{D, D''\}$, and then following a shortest path in G_θ between D'' and D'. Thus,

$$\delta_{G_\theta}(D, D') \leq dist(D, D'') + \delta_{G_\theta}(D'', D'). \tag{5}$$

2. In Lemma 7, we will show that

$$dist(D', D'') \leq dist(D, D') - f(\theta) \cdot dist(D, D''), \tag{6}$$

where $f(\theta) \sim 1 - \theta$.

3. Since $dist(D', D'') < dist(D, D')$, the induction hypothesis implies that

$$\delta_{G_\theta}(D', D'') \leq t \cdot dist(D', D'');$$

see Lemma 8. This inequality, together with (5), implies that

$$\delta_{G_\theta}(D, D') \leq dist(D, D'') + t \cdot dist(D'', D').$$

Thus, using (6), we obtain

$$\delta_{G_\theta}(D, D') \leq t \cdot dist(D, D') + (1 - t \cdot f(\theta)) \cdot dist(D, D'').$$

By taking $t = 1/f(\theta) \sim 1 + \theta$, we obtain the desired inequality (4), because $\{D, D'\}$ is a useful edge in T; see Lemma 9.

Observe that the above three steps are very similar to the steps made in the induction proof for bounding the stretch factors of the Yao graph (see Bose *et al.* [2, Lemma 10]), the Θ-graph (see Narasimhan and Smid [5, Section 4.1]), and the disk spanner of Bose *et al.* [1].

The main difficulty is in proving (6). Using the fact that both $radius(D')$ and $radius(D'')$ are at least equal to $radius(D)$, we will prove in Lemma 5 that

$$radius(D) \leq g(\theta) \cdot dist(D, D''),$$

where $g(\theta) \sim \theta$. Thus, $dist(D, D'')$ is approximately equal to the distance between $center(D)$ and the disk D''.

In the rest of this section, we will present the details. We start with a geometric lemma that will be needed in the proof of Lemma 5. The disk D_0 in the following lemma is not necessarily in the set $\mathcal{D}$. The lemma is illustrated in Figure 4.

Lemma 4 *Let D_0 be a disk in the plane, let c be a point outside D_0, let L be the line through c and $center(D_0)$, let L' be a line through c that is tangent to D_0, and let α be the angle between L and L'. Let ℓ be the distance between c and D_0, let y be the point that is common to D_0 and L', and let z be the point on L' for which $|cz| = \ell$. If $\alpha \leq \theta < \pi/2$, then*

$$radius(D_0) \leq \frac{\sin\theta}{1 - \sin\theta} \cdot \ell$$

and

$$|yz| \leq \frac{\sin\theta}{1 - \sin\theta} \cdot \ell.$$

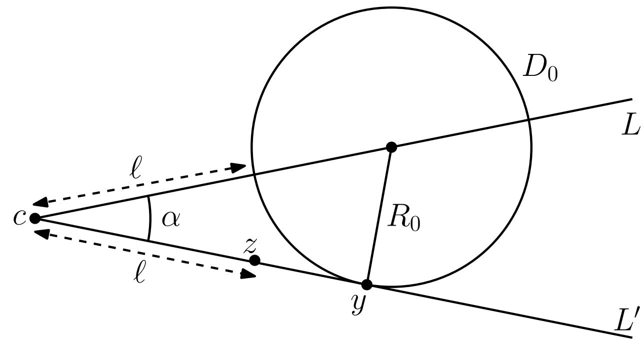

Figure 4: Illustrating Lemma 4.

Proof. Let $R_0 = radius(D_0)$. Since

$$\sin\alpha = \frac{R_0}{\ell + R_0},$$

we have

$$R_0 = \frac{\sin\alpha}{1-\sin\alpha} \cdot \ell. \tag{7}$$

The first inequality in the lemma follows because the function $\sin x/(1-\sin x)$ is increasing for $0 < x < \pi/2$.

To prove the second inequality, observe that

$$\cos\alpha = \frac{|cy|}{\ell + R_0}.$$

This, together with (7), implies that

$$|cy| = (\ell + R_0)\cos\alpha = \frac{\cos\alpha}{1-\sin\alpha} \cdot \ell$$

and, thus,

$$\begin{aligned}
|yz| &= |cy| - |cz| \\
&= \frac{\cos\alpha}{1-\sin\alpha} \cdot \ell - \ell \\
&= \frac{\cos\alpha + \sin\alpha - 1}{1-\sin\alpha} \cdot \ell \\
&\leq \frac{\sin\alpha}{1-\sin\alpha} \cdot \ell \\
&\leq \frac{\sin\theta}{1-\sin\theta} \cdot \ell.
\end{aligned}$$

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The following lemma states that the radius of the disk D is much smaller than the distance between D and D''.

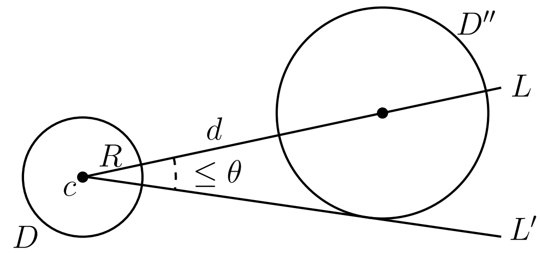

Figure 5: Illustrating Case 1 in the proof of Lemma 5.

Lemma 5 *Assuming that the cone angle θ is less than $\pi/6$,*

$$radius(D) \leq \frac{\sin\theta}{1 - 2\sin\theta} \cdot dist(D, D'').$$

Proof. Let $c = center(D)$, $R = radius(D)$, and $d = dist(D, D'')$. Let L be the line through c and $center(D'')$, let L' be a line through c that is tangent to D'', and let β be the angle between L and L'. We consider two cases.

Case 1: $\beta \leq \theta$. Refer to Figure 5.

It follows from the first inequality in Lemma 4 (with $D_0 = D''$, $\alpha = \beta$, and $\ell = R + d$) that

$$radius(D'') \leq \frac{\sin\theta}{1 - \sin\theta} \cdot (R + d).$$

Thus, since $R = radius(D) \leq radius(D'')$, we have

$$R \leq \frac{\sin\theta}{1 - \sin\theta} \cdot (R + d),$$

which implies that

$$R \leq \frac{\sin\theta}{1 - 2\sin\theta} \cdot d.$$

Case 2: $\beta > \theta$. Refer to Figure 6.

In this case, the first inequality in Lemma 4 cannot be applied to D''. In fact, $radius(D'')$ can be arbitrarily large. Below, we will show that, by applying the second inequality in Lemma 4 to D'', we get the desired upper bound on R.

Recall that C is the cone with apex c that contains $center(D'')$. We may assume, without loss of generality, that the cone bisector of C is horizontal and directed to the right, and the center of D'' is on or above this bisector.

Let x be the point on D'' that is closest to c. Let y_1 be the first point on D'' that is encountered when walking from c along the upper leg of C, and let z_1 be the point on this leg for which $|cz_1| = R + d$. Similarly, let y_2 be the first point on D'' that is encountered when walking from c along the lower leg of C, and let z_2 be the point on this leg for which $|cz_2| = R + d$. Observe that

$$|y_1 z_1| \leq |y_2 z_2|.$$

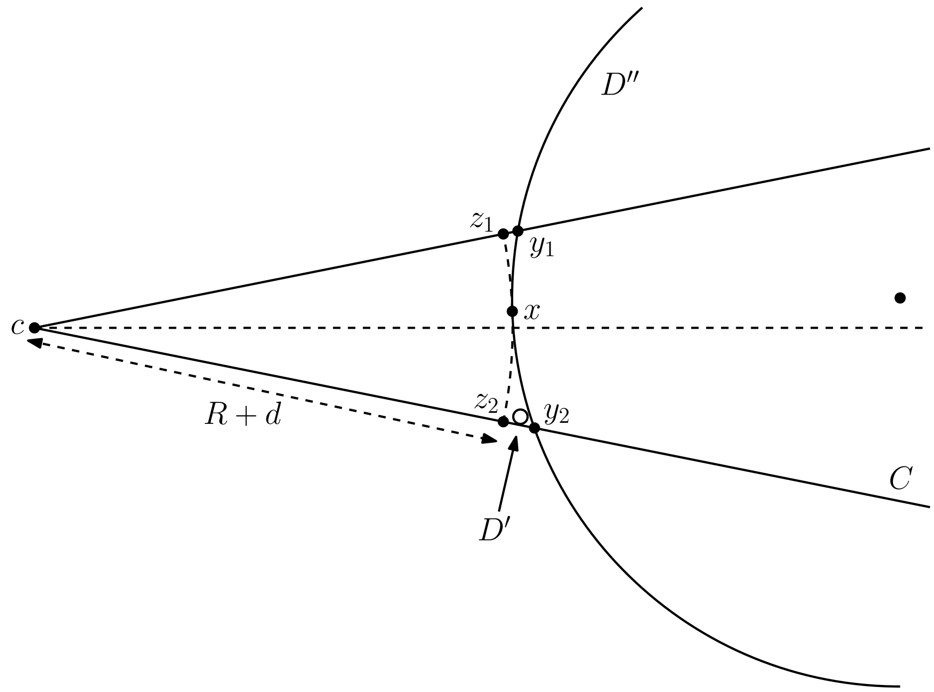

Figure 6: Illustrating Case 2 in the proof of Lemma 5. The disk D that is centered at c and whose radius is at most the radius of D', is not shown in the figure.

Recall that $dist(D, D'') \leq dist(D, D')$, D and D' are visible, and D, D' and D'' are pairwise disjoint. Therefore, the center of D' is either (i) in the "triangular" region bounded by x, y_1, and z_1, or (ii) in the "triangular" region bounded by x, y_2, and z_2. Assume that (i) holds. If we rotate D' in counterclockwise order about c, then it "moves" through the upper leg of C, entirely between the points y_1 and z_1. Thus, in this case,

$$radius(D') \leq |y_1 z_1|/2 \leq |y_2 z_2|.$$

Assume that (ii) holds. If we rotate D' in clockwise order about c, then it "moves" through the lower leg of C, entirely between the points y_2 and z_2. Thus, in this case,

$$radius(D') \leq |y_2 z_2|/2 \leq |y_2 z_2|.$$

Consider the following transformation: First, rotate D'' in counterclockwise order about c, until the center of D'' is on the upper leg of C. Since $\beta > \theta$, the lower leg of C intersects the interior of the rotated copy of D''. During this rotation, the value of $dist(D, D'')$ and the point z_2 do not change, whereas the distance $|y_2 z_2|$ increases.

After this rotation, we shrink D'', while keeping its center on the upper leg of C and keeping the value of $dist(D, D'')$ equal to d. We stop shrinking D'' as soon as it becomes tangent to the lower leg of C. During this shrinking, the distance $|y_2 z_2|$ increases.

Observe that the final disk D'' satisfies the conditions in Lemma 4: In this lemma, let D_0 be the final disk D'', let L and L' be the upper an lower legs of the cone C, respectively, let $\alpha = \theta$, and let $\ell = R + d$. Thus, by the second inequality in Lemma 4,

$$|y_2 z_2| \leq \frac{\sin\theta}{1 - \sin\theta} \cdot (R + d).$$

Since $R = \mathit{radius}(D) \leq \mathit{radius}(D')$, we conclude that

$$R \leq \mathit{radius}(D') \leq |y_2 z_2| \leq \frac{\sin\theta}{1 - \sin\theta} \cdot (R + d),$$

which implies that

$$R \leq \frac{\sin\theta}{1 - 2\sin\theta} \cdot d.$$

∎

The following lemma is due to Bose *et al.* [2].

Lemma 6 *Assume that $\theta < \pi/3$. Let a, b, and c be three distinct points in the plane. Assume that $|ca| \leq |cb|$ and the angle between the line segments ca and cb is at most θ. Then*

$$|ab| \leq |cb| - (1 - 2\sin(\theta/2)) \cdot |ca|.$$

Proof. Bose *et al.* [2, Lemma 10] prove that

$$|ab| \leq |cb| - \frac{2\cos\theta - 1}{1 + \sqrt{2 - 2\cos\theta}} \cdot |ca|.$$

Using the identity $\cos\theta = 1 - 2\sin^2(\theta/2)$, this is equivalent to the inequality in the lemma.
∎

The following lemma gives the main ingredient that will allow us to apply the induction hypothesis to the pair $\{D', D''\}$.

Lemma 7 *Assuming that the cone angle θ is less than 0.389 radians, the following two inequalities hold:*

$$\mathit{dist}(D', D'') \leq \mathit{dist}(D, D') - \left(1 - \frac{2(1 - \sin\theta)\sin(\theta/2)}{1 - 2\sin\theta}\right) \cdot \mathit{dist}(D, D'')$$

and

$$\mathit{dist}(D', D'') < \mathit{dist}(D, D').$$

Proof. Let $c = \mathit{center}(D)$ and $R = \mathit{radius}(D)$. Let a be the point on D'' that is closest to c, and let b be the point on D' that is closest to c. Refer to Figure 7. Observe that

$$\mathit{dist}(D', D'') \leq |ab|.$$

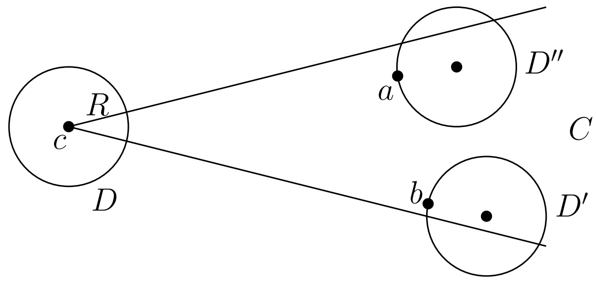

Figure 7: Illustrating the proof of Lemma 7.

Since

$$|ca| = R + dist(D, D'') \leq R + dist(D, D') = |cb|$$

and the angle between ca and cb is at most θ, we have, by Lemma 6,

$$\begin{aligned} |ab| &\leq |cb| - (1 - 2\sin(\theta/2)) \cdot |ca| \\ &= R + dist(D, D') - (1 - 2\sin(\theta/2))(R + dist(D, D'')) \\ &= dist(D, D') + 2R\sin(\theta/2) - (1 - 2\sin(\theta/2)) \cdot dist(D, D''). \end{aligned}$$

Lemma 5, which gives an upper bound on R, implies that

$$\begin{aligned} |ab| &\leq dist(D, D') + \left(\frac{2\sin\theta\sin(\theta/2)}{1 - 2\sin\theta} - (1 - 2\sin(\theta/2)) \right) \cdot dist(D, D'') \\ &= dist(D, D') - \left(1 - \frac{2(1 - \sin\theta)\sin(\theta/2)}{1 - 2\sin\theta} \right) \cdot dist(D, D''). \end{aligned}$$

This proves the first claim in the lemma. The second claim follows from the fact that, for $0 < \theta < 0.389$ radians,

$$\frac{2(1 - \sin\theta)\sin(\theta/2)}{1 - 2\sin\theta} < 1.$$

■

In the following lemma, we apply the induction hypothesis and obtain an upper bound on the shortest-path distance in G_θ between D' and D''.

Lemma 8 *Assuming that the cone angle* θ *is less than* 0.389 *radians,*

$$\delta_{G_\theta}(D', D'') \leq t \cdot dist(D', D'').$$

Proof. It follows from Lemmas 1 and 7 that

$$\delta_T(D', D'') \leq dist(D', D'') < dist(D, D').$$

Consider a shortest path in T between D' and D''. Each edge on this path is useful in T and has length less than $dist(D, D')$. Therefore, using the induction hypothesis,

$$\delta_{G_\theta}(D', D'') \leq t \cdot \delta_T(D', D''),$$

which is at most $t \cdot dist(D', D'')$. ∎

In the following, and final, lemma, we prove that (4) holds for the useful edge $\{D, D'\}$.

Lemma 9 *Assuming that the cone angle θ is less than 0.389 radians,*

$$\delta_{G_\theta}(D, D') \leq t \cdot \delta_T(D, D'),$$

where

$$t = \frac{1 - 2\sin\theta}{1 - 2\sin\theta - 2(1 - \sin\theta)\sin(\theta/2)}.$$

Proof. Recall that $\{D, D''\}$ is an edge of the graph G_θ. The path consisting of this edge, followed by a shortest path in G_θ between D'' and D', has length $dist(D, D'') + \delta_{G_\theta}(D'', D')$. Thus,

$$\delta_{G_\theta}(D, D') \leq dist(D, D'') + \delta_{G_\theta}(D'', D').$$

By first using Lemma 8 and then Lemma 7, we get

$$\begin{aligned}
\delta_{G_\theta}(D, D') &\leq dist(D, D'') + t \cdot dist(D'', D') \\
&\leq dist(D, D'') + t \left(dist(D, D') - \left(1 - \frac{2(1 - \sin\theta)\sin(\theta/2)}{1 - 2\sin\theta} \right) \cdot dist(D, D'') \right) \\
&= t \cdot dist(D, D') + \left(1 - t \left(1 - \frac{2(1 - \sin\theta)\sin(\theta/2)}{1 - 2\sin\theta} \right) \right) \cdot dist(D, D'').
\end{aligned}$$

Since $\delta_T(D, D') = dist(D, D')$, the lemma holds if

$$1 - t \left(1 - \frac{2(1 - \sin\theta)\sin(\theta/2)}{1 - 2\sin\theta} \right) \leq 0.$$

The latter inequality holds (with equality) for the value of t in the statement of the lemma. ∎

This concludes the analysis of the stretch factor of G_θ and, therefore, the proof of Theorem 1.

5 Concluding Remarks

We have revisited the problem of constructing spanners for pairwise disjoint disks. This problem can be generalized to any collection of n bounded and closed regions in the plane that are pairwise disjoint. In this case, the distance between any two regions R and R' is the length of the shortest line segment that connects R and R'. We leave as an open problem to decide whether a sparse spanner exists for any collection of regions. As a starting point, we suggest to study this problem for convex regions in the plane.

Acknowledgments

I would like to thank Pat Morin for mentioning the disk spanner problem to me a few years ago, and Frank Staals for reminding me of this problem.

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