On a family of strong geometric spanners that admit local routing strategies*

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Abstract

We introduce a family of directed geometric graphs, whose vertices are points in $\mathbb{R}^d$. The graphs G_λ^θ in this family depend on two real parameters λ and θ. For $\frac{1}{2} < \lambda < 1$ and $\frac{\pi}{3} < \theta < \frac{\pi}{2}$, the graph G_λ^θ is a strong t-spanner for $t = \frac{1}{(1-\lambda)\cos\theta}$. That is, for any two vertices p and q, G_λ^θ contains a path from p to q of length at most t times the Euclidean distance $|pq|$, and all edges on this path have length at most $|pq|$. The out-degree of any node in the graph G_λ^θ is $O(1/\phi^{d-1})$, where $\phi = \min(\theta, \arccos\frac{1}{2\lambda})$. We show that routing on G_λ^θ can be achieved locally. Finally, we show that all strong t-spanners are also t-spanners of the unit-disk graph.

1 Introduction

A graph G whose vertices are points in $\mathbb{R}^d$ and whose edges are line segments weighted by their Euclidean length is called a *geometric graph*. Such a graph G is called a *t-spanner* (for some real number $t \geq 1$) if for any pair p, q of vertices, the weight of the shortest path in G from p to q does not exceed $t|pq|$, where $|pq|$ denotes the Euclidean distance between p and q. The smallest t having this property is called the *stretch factor* or *spanning ratio* of the graph G. Any path from p to q in G whose length does not exceed $t|pq|$ is called a *t-spanning path*. A t-spanning path from p to q is called *strong* if the length of every edge in the path is at most $|pq|$. The graph G is called a *strong* t-spanner if there is a strong t-spanning path from any vertex to any other vertex.

The spanning properties of various geometric graphs have been studied extensively in the literature (see the book by Narasimhan and Smid [11] for a comprehensive survey on the topic). In this paper, we are interested in spanners that are defined by some proximity measure or emptiness criterion (see for example Bose *et al.* [2], Cardinal *et al.* [6]). Our work was initiated by Chávez *et al.* [7] who introduced a new geometric graph called *Half-Space Proximal* (*HSP*). Given a set P of points in $\mathbb{R}^2$, the geometric graph $HSP(P)$ with vertex set P is obtained by running the following algorithm for every point p of P:

1. Let $N(p)$ be the set $P \setminus \{p\}$.
2. Let r be a point in $N(p)$ that is closest to p.
3. Add the directed edge (p, r) to $HSP(P)$.

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4. Remove from $N(p)$ all points that are closer to r than to p, i.e., set

$$N(p) \leftarrow N(p) \setminus \{q \in N(p) : |qr| \leq |qp|\}.$$

5. If $N(p)$ is not empty, go to 2.

It follows from this algorithm that $HSP(P)$ contains an edge from a point p to a point q provided there is no point r in P such that (i) $|pr| \leq |pq|$, (ii) $HSP(P)$ contains an edge from p to r, and (iii) $|qr| \leq |qp|$.

The authors show that this graph has maximum out-degree at most 6; see Theorem 1 in [7]. The authors also present an outline of a proof that $HSP(P)$ has an upper bound of $2\pi + 1$ on its stretch factor; see Theorem 2 in [7]. To the best of our knowledge, however, no rigorous proof of the latter claim has appeared in the literature. Our attempts at finding a complete proof was the starting point of this work.

In Section 2, we introduce a family of directed geometric graphs. Each graph G_λ^θ in this family depends on two parameters λ and θ, with $\frac{1}{2} < \lambda < 1$ and $0 < \theta < \frac{\pi}{2}$. If λ and θ converge to 1 and $\pi/2$, respectively, the graph G_λ^θ converges to HSP. On the other hand, if λ converges to $\frac{1}{2}$, the graph G_λ^θ converges to a variant of Keil and Gutwin's Θ-graph; see [8]. In fact, for small values of θ and if λ is close to 1, the graph G_λ^θ is equal to a variant of the Yao graph; see [12].

We show that the maximum out-degree of G_λ^θ is bounded from above by a function that depends only on θ and λ. We also show that, for $\frac{1}{2} < \lambda < 1$ and $\frac{\pi}{3} < \theta < \frac{\pi}{2}$, G_λ^θ is a strong t-spanner, where t depends only on θ and λ. Furthermore, we show that graphs in this family admit local routing algorithms that find strong t-spanning paths. A routing algorithm is said to be *local* if the following holds: For any two vertices p and q, assume that we wish to construct a path from p to q. Also assume that a partial path from p to, say, x, has already been constructed. Then the next vertex of the path is obtained by using only the coordinates of the destination q and those points y for which (x, y) is a (directed) edge. (See [5] for a detailed description of this model). Local routing schemes are suitable for settings where it is infeasible for every node to know the entire graph or where the graph is constantly changing. When required to route without knowledge of the whole graph, routing schemes need to be simple schemes with the limited information available. Typical examples include: Greedy routing [5] (where a message is forwarded to the neighbor closest to the destination), Compass routing [9] (where a message is forwarded on the edge forming the smallest angle with the segment to the destination), Greedy-compass routing [3] (of the two neighbors straddling the line segment to the destination, the message is forwarded to the one closest to the destination) or other combinations (see [1] for a survey).

We show in Section 3 that all strong t-spanners are also spanners of the unit-disk graph, which are often used to model adhoc wireless networks (see the books Barbeau and Kranakis [1] and Li [10] for surveys of the area). Thus, by intersecting the graph G_λ^θ with the unit-disk graph, we obtain a spanner of the unit-disk graph.

2 The family of graphs G_λ^θ

In this section, we define the graph G_λ^θ and prove that it is a strong spanner of bounded out-degree. We first define this graph for point sets in the plane. At the end of this section, we show that the results are in fact valid for point sets in $\mathbb{R}^d$, for any dimension $d \geq 2$. We start by introducing some notation; see Figure 1.

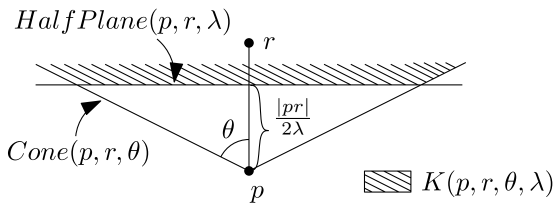


Figure 1: The destruction region $K(p, r, \theta, \lambda)$ of r with respect to p.

Definition 2.1 *Let p and r be two distinct points in the plane, and let λ and θ be positive real numbers.*

1. *We define $Cone(p, r, \theta)$ to be the cone of angle 2θ with apex p and having the line through p and r as its bisector.*

2. *We define $HalfPlane(p, r, \lambda)$ to be the half-plane containing r and having as its boundary the line perpendicular to $\overline{pr}$ and intersecting $\overline{pr}$ at distance $\frac{1}{2\lambda}|pr|$ from p.*

3. *The* destruction region $K(p, r, \theta, \lambda)$ *of r with respect to p is defined as*

$$K(p, r, \theta, \lambda) = Cone(p, r, \theta) \cap HalfPlane(p, r, \lambda).$$

Note that $HalfPlane(p, r, 1)$ is the halfplane containing r that is bounded by the perpendicular bisector of p and r. Also $K(p, r, \pi/2, 1)$ is equal to $HalfPlane(p, r, 1)$.

Let P be a finite set of points in the plane, and let λ and θ be real numbers with $\frac{1}{2} < \lambda < 1$ and $0 < \theta < \frac{\pi}{2}$. The directed graph $G_\lambda^\theta(P)$ is obtained by running the following algorithm for every point p in P:

1. Let $N(p)$ be the set $P \setminus \{p\}$.

2. Let r be a point in $N(p)$ that is closest to p.

3. Add the directed edge (p, r) to $G_\lambda^\theta(P)$.

4. Remove all points q in $K(p, r, \theta, \lambda)$ from $N(p)$, i.e., set

$$N(p) \leftarrow N(p) \setminus K(p, r, \theta, \lambda).$$

5. If $N(p)$ is not empty go to 2.

Notice that the point r in Step 2 is contained in $K(p, r, \theta, \lambda)$ and, hence, will be removed in Step 4. Therefore, since $|N(p)|$ is finite and at least one point is removed from $N(p)$ every time the algorithms executes steps 2 to 5, the algorithm terminates.

If λ and θ converge to 1 and $\pi/2$, respectively, the graph $G_\lambda^\theta(P)$ converges to $HSP(P)$. Moreover, if λ converges to $\frac{1}{2}$, the graph $G_\lambda^\theta(P)$ converges to a variant of the Θ-graph; see Keil and Gutwin [8]. Finally, if θ is sufficiently small, and λ is close to 1, the graph $G_\lambda^\theta(P)$ is equal to a variant of the Yao graph; see [12].

The following observation follows immediately from the algorithm.

Observation 2.2 *Let p and q be two distinct points of P and assume that (p, q) is not an edge in $G_\lambda^\theta(P)$. Then there exists a point r in $P \setminus \{p, q\}$ such that*

1. $|pr| \leq |pq|$,
2. (p, r) *is an edge in* $G_\lambda^\theta(P)$, *and*
3. $q \in K(p, r, \theta, \lambda)$.

Definition 2.3 *Let p and q be two distinct points of P and assume that (p, q) is not an edge in $G_\lambda^\theta(P)$. Any point r in $P \setminus \{q\}$ that satisfies the three conditions in Observation 2.2 is called a* destroyer *of the directed pair (p, q).*

2.1 Location of Destroyers

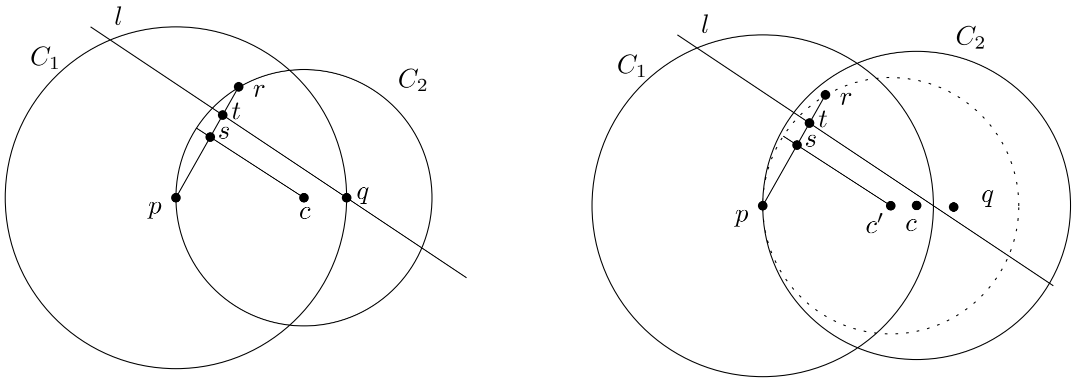
According to Observation 2.2, the directed pair (p, q) is not an edge in $G_\lambda^\theta(P)$, because of the existence of at least one point r acting as a destroyer. Given two distinct points p and q in P, where can such a point r lie? In this section, we define a region $\overline{K}(p, q, \theta, \lambda)$ and show that r must be in this region.

Definition 2.4 *Let p and q be two distinct points in the plane, and let λ and θ be real numbers with $\frac{1}{2} < \lambda < 1$ and $0 < \theta < \frac{\pi}{2}$.*

1. *We define $R(p, q, \lambda)$ to be the intersection of the disk centered at p with radius $|pq|$ and the disk centered at $p + \lambda(q - p)$ with radius $\lambda|pq|$.*
2. *We define $\overline{K}(p, q, \theta, \lambda)$ to be the intersection of $R(p, q, \lambda)$ and $Cone(p, q, \theta)$.*

Proposition 2.5 *If $q \in K(p, r, \theta, \lambda)$ and $|pr| \leq |pq|$, then $r \in R(p, q, \lambda)$.*

Proof: Let C_1 be the disk centered at p with radius $|pq|$, let $c = p + \lambda(q - p)$, and let C_2 be the disk centered at c with radius $\lambda|pq|$; see Figure 2. We have to show that $r \in C_1 \cap C_2$. Since $|pr| \leq |pq|$, r is contained in C_1. It remains to show that r is in C_2.



Case 1 where q lies on l — Case 2 where q does not lie on l

Figure 2: The location of a point r destroying the edge (p, q).

Let l be the boundary of $HalfPlane(p, r, \lambda)$, let s be the midpoint of $\overline{pr}$, let t be the intersection of l with $\overline{pr}$, and let c' be the intersection of $\overline{pq}$ with the bisector of $\overline{pr}$. We first consider the case when q lies on the line l. Since the triangles $\triangle ptq$ and $\triangle psc'$ are similar, we have

$$|pc'| = |pq|\frac{|ps|}{|pt|} = |pq|\frac{|pr|}{2|pt|} = |pq|\frac{\lambda|pr|}{|pr|} = \lambda|pq| = |pc|.$$

It follows that $c' = c$. Therefore, $|cr| = |cp|$, which proves that r is on the boundary of C_2.

In the case when q does not lie on l, we have $|pc'| < |pc|$ and r lies on the circle centered at c' going through p. Therefore, r is contained in C_2, which completes the proof. □

The following proposition follows immediately from the definition of $K(p, r, \theta, \lambda)$.

Proposition 2.6 *If $q \in K(p, r, \theta, \lambda)$, then $\angle qpr \leq \theta$.*

The following proposition states that any destroyer r of the directed pair (p, q) is contained in the region $\overline{K}(p, q, \theta, \lambda)$.

Proposition 2.7 *If $q \in K(p, r, \theta, \lambda)$ and $|pr| \leq |pq|$, then $r \in \overline{K}(p, q, \theta, \lambda)$.*

Proof: The proof follows from Propositions 2.5 and 2.6. □

2.2 The Stretch Factor of G_λ^θ

As observed earlier, for sufficiently small values of θ, the graph $G_\lambda^\theta(P)$ is equal to a variant of the Θ-graph or a variant of the Yao-graph, which are well-known to be spanners. Since the motivation of our work is to approximate $HSP(P)$, we will only consider the stretch factor of $G_\lambda^\theta(P)$ for values of θ which are larger than $\frac{\pi}{3}$. In this section, we will prove the following result:

Theorem 2.8 *Let P be a finite set of points in the plane, and let λ and θ be real numbers with $\frac{1}{2} < \lambda < 1$ and $\frac{\pi}{3} < \theta < \frac{\pi}{2}$. The directed graph $G_\lambda^\theta(P)$ is a strong t-spanner for $t = \frac{1}{(1-\lambda)\cos\theta}$.*

Notice that if λ converges to 1 and θ converges to $\pi/2$ (in this case $G_\lambda^\theta(P)$ converges to $HSP(P)$), the upper bound on the stretch factor of $G_\lambda^\theta(P)$ converges to infinity. On the other hand, if λ converges to $\frac{1}{2}$ and θ converges to $\frac{\pi}{3}$, then the upper bound on the stretch factor converges to 4.

In the rest of this section, we will prove Theorem 2.8. For any two distinct points p and q in P, let $d(p, q)$ denote the length of a shortest path from p to q in the graph $G_\lambda^\theta(P)$. We will show by induction in the sorted list of all pairwise Euclidean distances among points in P that $d(p, q) \leq t|pq|$. For ease of presentation, we assume that all these pairwise distances are distinct.

The base case of the induction is when p and q form the closest pair in P. In this case, the edge (p, q) is in $G_\lambda^\theta(P)$. Therefore, we have $d(p, q) = |pq| \leq t|pq|$.

From now on, we assume that p and q do not form a closest pair in P. Furthermore, we assume that $d(a, b) \leq t|ab|$ for any pair a and b of points in P for which $|ab| < |pq|$. We will show that $d(p, q) \leq t|pq|$.

If the edge (p, q) is in $G_\lambda^\theta(P)$, then $d(p, q) = |pq| \leq t|pq|$. It remains to consider the case when (p, q) is *not* an edge in $G_\lambda^\theta(P)$. Let r be an arbitrary destroyer of the pair (p, q). Notice that r exists by Observation 2.2. Thus, $|pr| < |pq|$, (p, r) is an edge in $G_\lambda^\theta(P)$, and $q \in K(p, r, \theta, \lambda)$.

Let C_1 be the disk centered at p with radius $|pq|$, let $c = p + \lambda(q - p)$, and let C_2 be the disk centered at c with radius $\lambda|pq|$. Recall that

$$\overline{K}(p, q, \theta, \lambda) = C_1 \cap C_2 \cap Cone(p, q, \theta).$$

By Proposition 2.7, we have $r \in \overline{K}(p, q, \theta, \lambda)$.

Since $r \in C_2$ and C_2 is contained in the radius-$|pq|$ disk centered at q, we have $|rq| < |pq|$. It follows that $d(r, q) \leq t|rq|$.

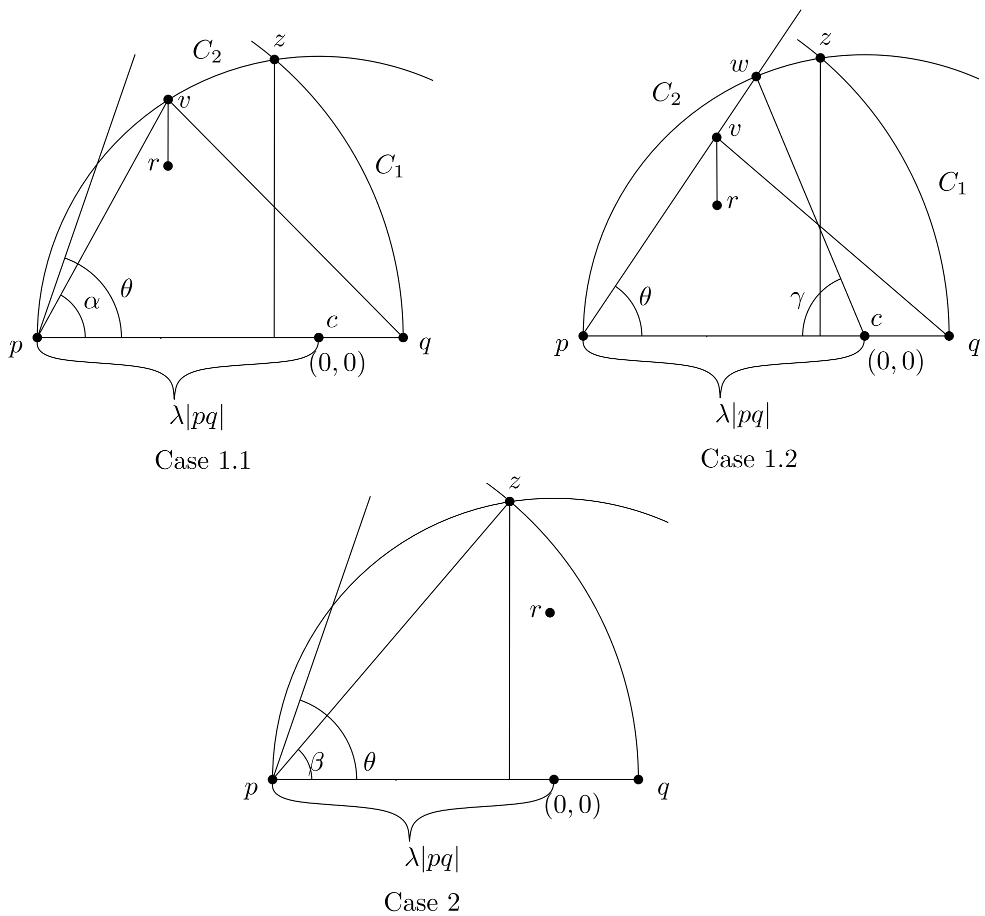


Figure 3: The cases for the proof of Theorem 2.8.

In the rest of the proof, we assume, without loss of generality, that c is the origin and both p and q are on the x-axis with p to the left of q; see Figure 3.

Let z be the intersection above the x-axis between the boundaries of the disks C_1 and C_2. We consider two cases, depending on whether or not $r_x \leq z_x$. (The notation a_x denotes the x-coordinate of a point a.)

Case 1: $r_x \leq z_x$.

Let v be the highest point in $\overline{K}(p, q, \theta, \lambda)$ having the same x-coordinate as r. We have:

$$d(p, q) \leq |pr| + d(r, q) \leq |pr| + t|rq| \leq |pv| + t|vq|.$$

Let $\alpha = \angle vpq$ and observe that $\alpha \leq \theta$, because $v \in Cone(p, q, \theta)$.

We consider two subcases, depending on whether v is on the boundary of C_2 or on the boundary of $Cone(p, q, \theta)$.

Case 1.1: v is on the boundary of C_2.

We will express $|pv|$ and $|vq|$ as functions of $\cos\alpha$. Consider the triangle $\triangle(pvc)$. Observe that $|vc| = |pc|$, because both v and p lie on the boundary of C_2, which is centered at c. We have

$$|pv| = 2\lambda|pq|\cos\alpha$$

and, by the law of cosines,

$$\begin{aligned}
|vq|^2 &= |pv|^2 + |pq|^2 - 2|pv||pq|\cos\alpha \\
&= 4\lambda^2|pq|^2\cos^2\alpha + |pq|^2 - 4\lambda|pq|^2\cos^2\alpha \\
&= |pq|^2(4\lambda^2\cos^2\alpha - 4\lambda\cos^2\alpha + 1).
\end{aligned}$$

Using $d(p, q) \leq |pv| + t|vq|$, this implies that

$$\begin{aligned}
d(p, q) &\leq 2\lambda|pq|\cos\alpha + t|pq|\sqrt{4\lambda^2\cos^2\alpha - 4\lambda\cos^2\alpha + 1} \\
&= |pq|\left(2\lambda\cos\alpha + t\sqrt{4\lambda^2\cos^2\alpha - 4\lambda\cos^2\alpha + 1}\right).
\end{aligned}$$

Thus, it suffices to show that

$$2\lambda\cos\alpha + t\sqrt{4\lambda^2\cos^2\alpha - 4\lambda\cos^2\alpha + 1} \leq t,$$

which can be rewritten as

$$t \geq \frac{2\lambda\cos\alpha}{1 - \sqrt{4\lambda^2\cos^2\alpha - 4\lambda\cos^2\alpha + 1}}.$$

Using the definition of t, it suffices to show that

$$\frac{1}{(1-\lambda)\cos\theta} \geq \frac{2\lambda\cos\alpha}{1 - \sqrt{4\lambda^2\cos^2\alpha - 4\lambda\cos^2\alpha + 1}}.$$

Since $\alpha \leq \theta < \pi/2$, we have $\cos\theta \leq \cos\alpha$. Therefore, it suffices to show that

$$\frac{1}{(1-\lambda)\cos\alpha} \geq \frac{2\lambda\cos\alpha}{1 - \sqrt{4\lambda^2\cos^2\alpha - 4\lambda\cos^2\alpha + 1}}.$$

The latter inequality can be verified by straightforward algebraic manipulation.

Case 1.2: v is on the boundary of $Cone(p, q, \theta)$.

In this case, we have $\alpha = \theta$. Let w be the intersection point above the x-axis between $Cone(p, q, \theta)$ and C_2, and let $\gamma = \angle pcw$. Since $|cp| = |cw| = \lambda|pq|$, we have $\gamma = \pi - 2\theta$. Observe that

$$|pv| \leq |pw| = 2\lambda\cos\theta|pq|.$$

As we have seen before, we have

$$d(p, q) \leq |pv| + t|vq|.$$

Thus, it suffices to show that $|pv| + t|vq| \leq t|pq|$, i.e.,

$$|vq|^2 \leq \left(|pq| - \frac{|pv|}{t}\right)^2.$$

By the law of cosines, we have

$$|vq|^2 = |pv|^2 + |pq|^2 - 2|pv||pq|\cos\theta,$$

which implies that it suffices to show that

$$|pv|^2 + |pq|^2 - 2|pv||pq|\cos\theta \leq \left(|pq| - \frac{|pv|}{t}\right)^2.$$

This inequality can be rewritten as

$$\left(1 - \frac{1}{t^2}\right)|pv| \leq \left(2\cos\theta - \frac{2}{t}\right)|pq|.$$

Since $|pv| \leq 2\lambda\cos\theta|pq|$, it suffices to show that

$$\left(1 - \frac{1}{t^2}\right)2\lambda\cos\theta \leq 2\cos\theta - \frac{2}{t}.$$

Using the definition of t, we obtain $2\cos\theta - \frac{2}{t} = 2\lambda\cos\theta$. Therefore, it suffices to show that

$$\left(1 - \frac{1}{t^2}\right)2\lambda\cos\theta \leq 2\lambda\cos\theta,$$

which obviously holds.

Case 2: $r_x > z_x$**.**

Let $\beta = \angle zpq$. We first compute the value of $\cos\beta$. From the definitions of C_1 and C_2, we have

$$z_x^2 + z_y^2 = \lambda^2|pq|^2$$

and

$$(z_x - p_x)^2 + z_y^2 = |pq|^2.$$

Therefore, since $p_x = -\lambda|pq|$, we have

$$z_x = \frac{|pq|(1 - 2\lambda^2)}{2\lambda},$$

which implies that

$$\cos\beta = \frac{\lambda|pq| + z_x}{|pq|} = \lambda + \frac{1 - 2\lambda^2}{2\lambda} = \frac{1}{2\lambda}.$$

Since $\lambda < 1$, it follows that $\cos\beta > \frac{1}{2}$, which implies that $\beta < \frac{\pi}{3}$. Thus, since $\theta > \frac{\pi}{3}$, we have $\beta < \theta$. We have

$$d(p,q) \leq |pr| + d(r,q) \leq |pr| + t|rq| \leq |pz| + t|zq| = |pq| + t|zq|.$$

By the law of cosines in $\triangle zpq$, and using the fact that $|pz| = |pq|$, we have

$$|zq|^2 = \left(2 - \frac{1}{\lambda}\right)|pq|^2,$$

which implies that

$$d(p,q) \leq |pq| + t|pq|\sqrt{2 - \frac{1}{\lambda}}.$$

Therefore, it suffices to show that

$$t \geq \frac{1}{1 - \sqrt{2 - \frac{1}{\lambda}}}.$$

Since $0 < \beta < \theta < \pi/2$, we have $\cos\beta > \cos\theta$, and thus

$$\begin{aligned} t &= \frac{1}{(1-\lambda)\cos\theta} \\ &\geq \frac{1}{(1-\lambda)\cos\beta} \\ &= \frac{1}{(1-\lambda)(1/2\lambda)} \\ &= \frac{2\lambda}{1-\lambda} \\ &\geq \frac{1}{1 - \sqrt{2 - \frac{1}{\lambda}}}, \end{aligned}$$

where the last inequality holds because it is equivalent to $(1-\lambda)^2 \geq 0$.

This completes the proof of the claim that $G_\lambda^\theta(P)$ is a t-spanner. In fact, since both $|pr|$ and $|rq|$ are shorter than $|pq|$, it follows from this proof that G_λ^θ is in fact a strong t-spanner. Thus, we have completed the proof of Theorem 2.8.

Observe that our proof of Theorem 2.8 provides a simple local routing algorithm: To find a path from p to q, if the edge (p,q) is in $G_\lambda^\theta(P)$, then take the edge. Otherwise, take an arbitrary edge (p,r) in $G_\lambda^\theta(P)$ such that r is a destroyer of the pair (p,q). By considering all outgoing edges of p and using Definition 2.3, a destroyer r can be found solely from the positions of p and q. In other words, determining which of the outgoing neighbors of p in $G_\lambda^\theta(P)$ destroyed the pair (p,q) is a local computation.

2.3 G_λ^θ in $\mathbb{R}^d$

The definition of the graph G_λ^θ extends to $\mathbb{R}^d$ for any $d > 2$ in a natural way. In this case, the cone in Definition 2.1 is a d-dimensional cone, and the half-plane in Definition 2.1 is a d-dimensional half-space. Moreover, the statement and the proof of Theorem 2.8 are exactly the same: Let P be a set of points in $\mathbb{R}^d$, and let p and q be two points in P. If the edge (p,q) is in G_λ^θ, then we are done. Otherwise, there exists a point r in P that is destroying the directed pair (p,q). Let Π be a two-dimensional plane that contains p, q, and r. The proof above applies in the plane Π. Therefore, we have the following corollary:

Corollary 2.9 *Let $d \geq 2$, let P be a finite set of points in $\mathbb{R}^d$, and let λ and θ be real numbers with $\frac{1}{2} < \lambda < 1$ and $\frac{\pi}{3} < \theta < \frac{\pi}{2}$. The directed graph $G_\lambda^\theta(P)$ is a strong t-spanner for $t = \frac{1}{(1-\lambda)\cos\theta}$.*

2.4 G_λ^θ is of Bounded Out-Degree

In this section, we prove the following result:

Theorem 2.10 *Let P be a finite set of points in the plane, and let λ and θ be real numbers with $\frac{1}{2} < \lambda < 1$ and $0 < \theta < \frac{\pi}{2}$. The out-degree of any point of P in the graph $G_\lambda^\theta(P)$ is at most $\lfloor 2\pi / \min(\theta, \arccos \frac{1}{2\lambda}) \rfloor$.*

Proof: Let p be a point of P, and consider two edges (p, r) and (p, s) in the graph $G_\lambda^\theta(P)$. We may assume without loss of generality that $|pr| \leq |ps|$. Let l be the line perpendicular to $\overline{pr}$ through $p + \frac{1}{2\lambda}(r - p)$; see Figure 4. Then either $\angle spr \geq \theta$ or s lies on the same side of l as p. In the latter case, the angle $\angle spr$ is at least $\arccos \frac{1}{2\lambda}$. The angle $\angle spr$ is thus at least $\min(\theta, \arccos \frac{1}{2\lambda})$, which means that p has at most $\lfloor 2\pi / \min(\theta, \arccos \frac{1}{2\lambda}) \rfloor$ outgoing edges. □

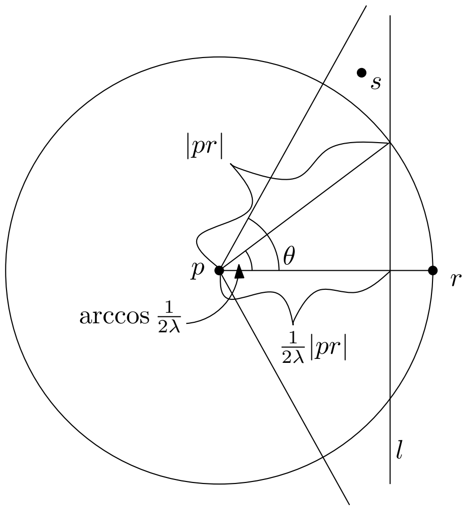


Figure 4: The G_λ^θ graph has bounded out-degree.

Corollary 2.11 *If P is a finite set of points in the plane, $\theta \geq \pi/3$ and $\lambda > \frac{1}{2\cos(2\pi/7)}$, then the out-degree of any point of P in the graph $G_\lambda^\theta(P)$ is at most six.*

In higher dimensions, the maximum out-degree of the graph G_λ^θ depends not only on λ and θ, but also on the dimension d. The key observation that has to be made is that any two edges that are outgoing from the same point still form an angle of at least $\min(\theta, \arccos \frac{1}{2\lambda})$. Therefore, Theorem 2.12 below provides an upper bound on the maximum out-degree of the graph G_λ^θ defined on points in $\mathbb{R}^d$.

Theorem 2.12 [1] *Let $d \geq 2$ be an integer constant, let ϕ be a real number such that $0 < \phi < \pi$, and let S be a set of points in $\mathbb{R}^d \setminus \{0\}$ such that* $\text{angle}(q, r) > \phi$ *for any two distinct points q and r in S. Then the size of S is $O(1/\phi^{d-1})$.*

Corollary 2.13 *Let P be a finite set of points in $\mathbb{R}^d$, and let λ and θ be real numbers with $\frac{1}{2} < \lambda < 1$ and $0 < \theta < \frac{\pi}{2}$. The out-degree of any point of P in the graph $G_\lambda^\theta(P)$ is $O(1/\phi^{d-1})$, where $\phi = \min(\theta, \arccos \frac{1}{2\lambda})$.*

3 Unit-Disk Graph Spanners

In Section 2, we showed that the graph $G_\lambda^\theta(P)$, where P is a set of points in $\mathbb{R}^d$, is a strong t-spanner of the complete graph of these points, for $t = \frac{1}{(1-\lambda)\cos\theta}$. In this section, we show that strong t-spanners lead to t-spanners of the unit-disk graph. That is, the length of a shortest path in the graph resulting from intersecting a strong t-spanner with the unit-disk graph is at most t times the length of a shortest path between the points in the unit-disk graph. Before proving this claim, we introduce some notation.

For simplicity of exposition, we will assume that P is a set of points in $\mathbb{R}^d$, such that no two pairs of points are at equal distance from each other. The *complete geometric graph* $C(P)$ is the graph whose vertex set is P and whose edge set is the set of all unordered pairs of distinct points in P. Each edge in this graph has a weight equal to the Euclidean distance between its vertices. Let $e_1, \ldots, e_{\binom{n}{2}}$ be the edges of $C(P)$ sorted according to their lengths $L_1, \ldots, L_{\binom{n}{2}}$. For $i = 1, \ldots, \binom{n}{2}$, we denote by $C_i(P)$ the geometric graph consisting of all edges whose length is no more than L_i. In general, for any graph G whose vertex set is V, we define G_i to be the graph $G \cap C_i(V)$. Let $UDG(P)$ be the unit-disk graph of P, which is the graph whose vertex set is P and with edges between pairs of vertices whose distance is not more than one. Note that $UDG(P) = C_i(P)$ for some i.

We now show the relationship between strong t-spanners and unit-disk graphs.

Observation 3.1 *If S is a strong t-spanner of $C(P)$, then for all $i = 1, \ldots, \binom{n}{2}$ and for all $j \leq i$, the graph S_i contains a t-spanning path linking the vertices incident to e_j.*

Proof: Let p and q be the vertices incident to e_j. Consider a strong t-spanning path in S between p and q. Each edge on this path has length at most $|pq| = L_j \leq L_i$. Therefore, each edge is in S_i. □

Proposition 3.2 *If S is a strong t-spanner of $C(P)$, then for all $i = 1, \ldots, \binom{n}{2}$, the graph S_i is a t-spanner of $C_i(P)$.*

Proof: Let a and b be any two points such that the shortest-path distance $d_{C_i(P)}(a, b)$ is finite. We need to show that there exists a path in S_i between a and b whose length is at most $t \cdot d_{C_i(P)}(a, b)$. Let $a = p_1, p_2, \ldots, p_k = b$ be a shortest path in $C_i(P)$ between a and b, so that

$$d_{C_i(P)}(a, b) = \sum_{j=1}^{k-1} |p_j p_{j+1}|.$$

[1] Theorem 5.3.1 of [11], where 0 is the origin and $\text{angle}(q, r)$ is the angle between q and r with the origin as apex.

By Observation 3.1, for each edge (p_j, p_{j+1}) there is a path in S_i between p_j and p_{j+1} whose length is at most $t \cdot |p_j p_{j+1}|$. It follows that

$$d_{S_i(P)}(a,b) \leq \sum_{j=1}^{k-1} t \cdot |p_j p_{j+1}| = t \sum_{j=1}^{k-1} |p_j p_{j+1}| = t \cdot d_{C_i(P)}(a,b).$$

□

Corollary 3.3 *If S is a strong t-spanner of $C(P)$, then $S \cap UDG(P)$ is a strong t-spanner of $UDG(P)$.*

Proof: The proof follows from Proposition 3.2 and the observation that $UDG = C_i$ for some i. □

Thus, we have shown a sufficient condition for a graph to be a spanner of the unit-disk graph. We now show that this condition is also necessary.

Proposition 3.4 *If S is a subgraph of $C(P)$ such that for all $i = 1, \ldots, \binom{n}{2}$, the graph S_i is a t-spanner of $C_i(P)$, then S is a strong t-spanner of $C(P)$.*

Proof: Let a, b be any pair of points in P. We have to show that in S, there is a path between a and b such that (i) the length of this path is at most $t \cdot |ab|$ and (ii) every edge on this path has length at most $|ab|$.

Let i be the index such that $e_i = (a, b)$. We know that S_i is a t-spanner of $C_i(P)$. Since $C_i(P)$ contains e_i, we have $d_{C_i(P)}(a,b) = |ab|$. Hence, there is a path in S_i (and therefore in S) whose length is at most $t \cdot d_{C_i(P)}(a,b) = t|ab|$. Also, since this path is in S_i, all of its edges have length at most $L_i = |ab|$. □

The two last results allow us to determine whether or not given families of geometric graphs give rise to spanners of the unit-disk graph. Below, we give some examples. We have seen that the graph $G_\lambda^\theta(P)$ is a strong t-spanner. Therefore, the intersection of $G_\lambda^\theta(P)$ with the unit-disk graph $UDG(P)$ is a spanner of $UDG(P)$. In Bose *et al.* [4], it is shown that both the Yao graph (see [12]) and the Delaunay triangulation are strong t-spanners for some constant t. Therefore, by intersecting any of these graphs with $UDG(P)$, we obtain spanners of $UDG(P)$. Finally, the intersection of the directed Θ-graph (see [8]) with $UDG(P)$ is not necessarily a spanner of $UDG(P)$. This claim follows from the fact that, in some cone, the edge that is chosen may not be the shortest edge. Hence, the path from a point p to a point q may contain edges whose lengths are larger than $|pq|$. An example is given in Figure 5. In this example, $|pq| \leq 1$, whereas $|pr| > 1$. The directed Θ-graph contains the directed edge (p, r), which is not an edge in the unit-disk graph. Thus, the intersection between the directed Θ-graph with UDG may not even be strongly connected.

4 Open Problems

We have introduced a new family of directed geometric graphs. Each graph G_λ^θ in this family has bounded out-degree and, for $\frac{1}{2} < \lambda < 1$ and $\frac{\pi}{3} < \theta < \frac{\pi}{2}$, is a strong t-spanner for $t = \frac{1}{(1-\lambda)\cos\theta}$. We leave it as an open problem to determine the smallest t for which G_λ^θ is a t-spanner. We also leave

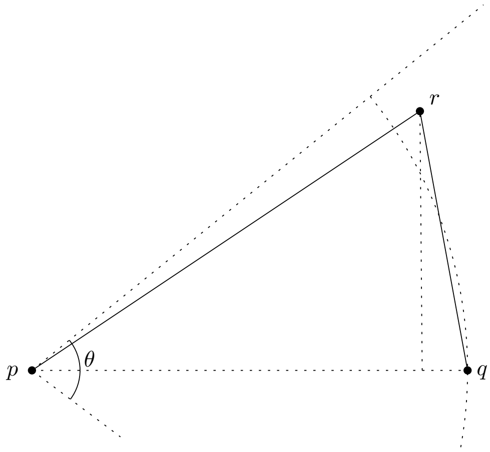


Figure 5: The Θ-graph is not a strong t-spanner.

open the problem of designing an efficient algorithm for computing the graph G^θ_λ. In particular, can this graph be computed in close to linear time?

We have shown that a t-spanning path in G^θ_λ can be computed by a local algorithm. We have also shown that the intersection of G^θ_λ with the unit-disk graph UDG is a t-spanner of UDG. We leave it as an open problem to design a local routing algorithm for the intersection of G^θ_λ with UDG.

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