CONSIDER GENERAL NON LINEAR SCHEME:

$$p_i^{(n+1)} = p_i - g_i \qquad \alpha_j\,,\ \beta = 0$$

$$= p_i + h_i \qquad \alpha_j\,,\ \beta = 1$$

$$= p_i + \sum_{j \neq i} g_j \qquad \alpha_i\,,\ \beta = 0$$

$$= p_i - \sum_{j \neq i} h_j \qquad \alpha_i\,,\ \beta = 1.$$

WHAT ARE CONDITIONS FOR THIS SCHEME TO BE

ABS. EXPEDIENT ?

WHAT IS $\Delta p_i(n) = E\left[p_i(n+1) - p_i(n) \mid \underline{p} \right]$

$$
\begin{aligned}
p_i(n+1) - p_i(n) &= -g_i && \text{w.p. } \sum_{j \neq i}{}' p_j (1 - c_j) \\
&= \sum_{j \neq i}{}' g_j && \text{w.p. } p_i (1 - c_i) \\
&= h_i && \text{w.p. } \sum_{j \neq i}{}' p_j c_j \\
&= -\sum_{j \neq i}{}' h_j && \text{w.p. } p_i c_i
\end{aligned}
$$

$\therefore \ \Delta p_i(n) = E\left[p_i(n+1) - p_i(n) \mid \underline{p} \right]$

$$
\begin{aligned}
&= p_i \sum_{j \neq i}{}' g_j \ (3) - g_i \sum_{j \neq i}{}' p_j \ (4) \\
&\quad - c_i p_i \left[\sum_{j \neq i}{}' (g_j + h_j) \right] \ (2) \\
&\quad + (g_i + h_i) \left[\sum_{j \neq i} c_j p_j \right] \ (1)
\end{aligned}
$$

$\Rightarrow$

$p_i \sum_{j=1}^{R} g_j$ (3) $- g_i$ (4)

$-$ ~~~~~~ (2)

$+$ ~~~~~~ (1)

(3) & (4) $\quad p_i \sum_{j=1}^{R} g_j - g_i \quad$ since $\sum_{j \neq i} p_j = 1 - p_i$

To get $\Delta M(n) = \sum_{i=1}^{R} c_i \, \Delta p_i$

DO ONE MORE SUMMATION.

$$\Delta M(n) = \sum_{i=1}^{R} \sum_{j \neq i} c_i c_j p_j \left[g_i + h_i \right] \quad - ①$$

$$- \sum_{i=1}^{R} \sum_{j \neq i} c_i^2 p_i \left[g_j + h_j \right] \quad - ②$$

$$+ \sum_{i=1}^{R} \sum_{j=1}^{R} c_i p_i g_j \quad - ③'$$

$$- \sum_{i=1}^{R} c_i g_i \quad - ④'$$

CONCENTRATE ON c's.

Let $\underline{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_R \end{bmatrix}$ THEN, WE HAVE QUADRATIC FORM

$$\Delta M(n) = \underline{c}^T J \underline{c} + \underline{c}^T \underline{b}$$

where $J_{i,j} = p_j (g_i + h_i)$

$$J_{i,i} = -p_i \sum_{j \neq i} (g_j + h_j)$$ note summation.

$$\Delta M(n) = \underline{c}^T J \underline{c} + \underline{c}^T \underline{b}$$

$$J_{i,j} = p_j (g_i + h_i)$$

$$J_{i,i} = -p_i \left(\sum_{j \neq i}' (g_j + h_j) \right)$$

note summation.

AND

$$\underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_R \end{bmatrix}$$

$$b_i = p_i \sum_{j=1}^{R} g_j - g_i$$

USING THIS - MANY RESUTS CAN BE PROVED.

$$\boxed{\Delta M(n) = \underline{c}^T J \underline{c} + \underline{c}^T \underline{b}}$$

A WORD ON QUADRATIC FORMS.

$$[x_1 \;\; x_2] \begin{bmatrix} 4 & 3 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [4x_1 + 4x_2, \; 3x_1 + 7x_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 4x_1^2 + 7x_1 x_2 + 7x_2^2 . = \sum_{i=1}^{R} \sum_{j=1}^{R} J_{i,j} x_i x_j$$

$$\therefore \; 4x_1^2 + 7x_1 x_2 + 7x_2^2$$

$$= \underline{x}^T \begin{bmatrix} 4 & 3.5 \\ 3.5 & 7 \end{bmatrix} x$$ <u>gives same answer.</u>

Diagonal part - coeff. of $x_i^2 = J_{i,i}$

Off diagonal - $J_{i,j} = \frac{\text{coeff. } x_i x_j}{2}$

for both.

$\Rightarrow$ <u>symmetric form.</u>

THEOREM

A GENERAL NONLINEAR SCHEME is Abs. Exp.

IF AND ONLY IF:

$$\frac{g_1(\underline{p})}{p_1} = \frac{g_2(\underline{p})}{p_2} = \cdots = \frac{g_R(\underline{p})}{p_R} = \lambda(\cdot)$$

$$\frac{h_1(\underline{p})}{p_1} = \frac{h_2(\underline{p})}{p_2} = \cdots = \frac{h_R(\underline{p})}{p_R} = \mu(\cdot)$$

(SYMMETRY CONDITIONS)

PROOF.

SUFFICIENCY

GIVEN THE SYMM. COND. - PROVE ABS. EXP.

$$\frac{g_j}{p_j} = \lambda \quad , \quad \frac{h_j}{p_j} = \mu.$$

THEN

$$J_{i,j} = p_j(g_i + h_i)$$

$$= p_j(p_i\lambda + p_i\mu) = p_i p_j(\lambda + \mu)$$

$$J_{i,i} = -p_i\Big(\sum_{j\neq i} g_j + h_j\Big)$$

$$= -p_i \sum_{j\neq i} p_j(\lambda+\mu) = -p_i(1-p_i)(\lambda+\mu)$$

$$b_i = p_i \sum_{j=1}^{R} g_j - g_i$$

$$= p_i \sum_{j=1}^{R} (p_j)\lambda - g_i$$

$$= p_i \lambda \sum_{j=1}^{R} p_j - g_i = g_i - g_i = 0.$$

$$\therefore \quad \underline{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

<u>HENCE</u> (see later)

$$\Delta M(n) = \underline{c}^T J \underline{c} + \cancelto{0}{\underline{c}^T \underline{b}}$$ (other part goes to zero)

$$= (\lambda + \mu) \sum_{i=1}^{R} \sum_{j \neq i} (c_i c_j - c_i^2)\, p_i p_j$$ (see later)

THUS

$$J = (\lambda + \mu) \begin{bmatrix} -p_1(1-p_1) & p_1 p_2 & \cdots & p_1 p_R \\ p_2 p_1 & -p_2(1-p_2) & \cdots & p_2 p_R \\ & & & \\ p_R p_1 & p_R p_2 & \cdots & -p_R(1-p_R) \end{bmatrix}$$

OR

$$\Delta M(n) = \begin{bmatrix} c_1 & c_2 & \cdots & c_R \end{bmatrix} \begin{bmatrix} & & \\ & J & \\ & & \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_R \end{bmatrix}$$

from $(1-p_i)$ term

$$\Rightarrow \Delta M(n) = (\lambda + \mu) \sum_{i=1}^{R} \sum_{j \neq i} \left(c_i c_j - c_i^2 \right) p_i p_j$$

WRITE THIS AS QUADRATIC FORM IN $\underline{P}$

i.e.

$$\Delta M(n) = (\lambda + \mu)\, \underline{P}^T \, \mathbb{C} \, \underline{P}$$

where $\underline{P} = \begin{bmatrix} p_1 \\ \vdots \\ p_R \end{bmatrix}$ and $\mathbb{C} = \begin{bmatrix} & & \\ & \cdot & \\ & & \end{bmatrix}$

$$\mathbb{C}_{i,i} = 0$$

$$C_{i,j} = \frac{(C_i C_j - C_i^2) + (C_j C_i - C_j^2)}{2} = -\frac{(C_i - C_j)^2}{2}.$$

$$= \begin{bmatrix} \ddots & * \\ * & \ddots \end{bmatrix}$$

Every component in $*$ is -ve.

HENCE

$$\Delta M(n) = \underline{p}^T \, \mathbb{C} \, \underline{p}$$

$$= + \left(\frac{\lambda + \mu}{2}\right) \sum_{i=1}^{R} \sum_{i \neq j} \left[-(C_i - C_j)^2 \right] p_i p_j$$

which is always -ve.

HENCE $\Delta M(n) < 0 \Rightarrow$ ABS. EXPEDIENCY.

NECESSITY

ABS. EXPEDIENCY $\Rightarrow$ Symmetry Conditions.

OBSERVE ABS. EXPEDIENCY MEANS

$$\Delta M(n) < 0$$

$\Delta M(n) \equiv 0$ if $c_1 = c_2 = \cdots = c_R$

since $$\Delta M(n) = \sum_{i=1}^{R} c_i \, \Delta p_i$$

$$= c \sum_{i=1}^{R} \Delta p_i = 0$$

$\therefore$ $\Delta M(n)$ has maximum at $c_1 = c_2 = \cdots c_R = 0$

if Scheme is Absolutely Expedient.

i.e. $$\frac{\partial(\Delta M(n))}{\partial c_k} \equiv 0 \qquad k = 1, .. R$$

But $$\Delta M(n) = \underline{c}^T J \underline{c} + \underline{b}^T c$$

$$\therefore \frac{\partial(\Delta M(n))}{\partial c_k} = (J + J^T)\underline{c} + \underline{b} = 0$$

Since $c_1 = c_2 = \cdots c_R = 0$ is maximum, k-th eqn.

$$\underbrace{c\left[(g_k+h_k) - p_k \textstyle\sum'(g_j+h_j)\right]}_{x} + \underbrace{p_k \textstyle\sum' g_j - g_k}_{y} = 0$$

Valid for any 'c'.

$xc + y = 0$ for all $c \Rightarrow$ $x = 0$
$y = 0$

$y = 0$

$$p_k \textstyle\sum' g_j - g_k = 0$$

i.e. $\dfrac{g_k}{p_k} = \sum' g_j$

or $\boxed{\dfrac{g_1}{p_1} = \dfrac{g_2}{p_2} = \cdots \dfrac{g_R}{p_R} = \sum' g_j = \lambda}$

$x = 0$ (substitute $g_K = p_K \sum' g_j$)

yields

$$h_K - p_K \sum_{j=1}^{R} h_j = 0 \Rightarrow \frac{h_K}{p_K} = \sum_{j=1}^{R} h_j$$

or $\boxed{\dfrac{h_1}{p_1} = \dfrac{h_2}{p_2} = \cdots \dfrac{h_R}{p_R} = \sum' h_j = \mu}$

AND THEOREM IS PROVED.

$$(J + J^T)\underline{c} + \underline{b} = 0$$

1st row.

$$\Big(-\cancel{2} p_1 \sum_{j\neq 1} (g_j + h_j) + p_2(g_1+h_1) + p_3(g_1+h_1) + p_4(g_1+h_1) + \dots p_R(g_1+h_1)$$
$$+ p_1(\cancel{g_2+h_2}) + p_1(\cancel{g_3+h_3}) + p_1(\cancel{g_4+h_4}) + \dots p_1(\cancel{g_R+h_R}) \Big) c + b_1 = 0$$

$$\Rightarrow (g_1+h_1)\sum_{j\neq 1} p_j - p_1 \sum_{j\neq 1}(g_j+h_j) + \Big(p_1\sum_{j=1}^{R} g_j - g_1\Big) = 0.$$

$$\Rightarrow \Big[(g_1+h_1)(1-p_1) - p_1 \sum_{j\neq 1}(g_j+h_j)\Big]c + \Big(p_1\sum_{j=1}^{R} g_j - g_1\Big) = 0$$

$$\Rightarrow \Big[(g_1+h_1) - p_1 \sum_{j=1}^{R}(g_j+h_j)\Big]c. + \Big(p_1\sum_{j=1}^{R} g_j - g_1\Big) = 0.$$

yields sol^n.

$$
J = \begin{bmatrix}
-p_1 \sum_{j \neq 1} (g_j + h_j) & p_2 (g_1 + h_1) & p_3 (g_1 + h_1) & \cdots & p_R (g_1 + h_1) \\
p_1 (g_2 + h_2) & -p_2 \sum_{j \neq 2} (g_j + h_j) & p_3 (g_2 + h_2) & & p_R (g_2 + h_2) \\
p_1 (g_3 + h_3) & & \ddots & & \\
\vdots & & & \ddots & \\
p_1 (g_R + h_R) & & & & -p_R \sum_{j \neq R} (g_j + h_j)
\end{bmatrix}
$$

$$
J^T = \begin{bmatrix}
-p_1 \sum_{j \neq 1} (g_j + h_j) & p_1 (g_2 + h_2) & p_1 (g_3 + h_3) & \cdots & p_R (g_R + h_R) \\
 & & & & \\
 & & & & \\
\end{bmatrix}
$$

CONSEQUENCES OF THEOREM.

iff

$$\frac{g_1(\underline{p})}{p_1} = \cdots = \frac{g_R(\underline{p})}{p_R} = \lambda$$

$$\frac{h_1(\underline{p})}{p_1} = \cdots = \frac{h_R(\underline{p})}{p_R} = \mu$$

EVERY ABS. EXP. SCHEME CAN BE WRITTEN IN FORM:

$$\begin{aligned} p_i(n+1) &= p_i(1-\lambda) && \alpha_j, \ \beta = 0 \\ &= p_i(1+\mu) && \alpha_j, \ \beta = 1 \\ &= p_i + \lambda(1-p_i) && \alpha_i, \ \beta = 0 \\ &= p_i - \mu(1-p_i) && \alpha_i, \ \beta = 1 \end{aligned}$$

Since

$$\left.\begin{aligned} 0 &< g_j < p_j \\ \text{and} \quad 0 &< \sum_{j \neq i}{}' (p_j + h_j) \leq 1 \end{aligned}\right\} \Longrightarrow$$

$$0 < \lambda(\underline{p}) < 1$$

$$0 < \mu(\underline{p}) < \min_i \frac{p_i}{1-p_i}, \quad p_i \in (0,1)$$